Supplement: Lie Algebras

Reminder: For a 1-parameter Lie group with elements g

and $g(\tau+\varepsilon) = g(\tau)g(\varepsilon)$

we had for a representation

$$D(g(\varepsilon)) = 1 - i\,\varepsilon T + \mathcal{O}(\varepsilon^2)$$

T = generator

Finite transformations are obtained by exponentiation:

$$D(g(\varepsilon)) = e^{-i\varepsilon T}$$

Now we are interested in a Lie group with several parameters ε^a. There are functions $f^a(\varepsilon,\tau)$ such that

$$g(\varepsilon)\,g(\tau) = g(f(\varepsilon,\tau))$$

Choose

$$g(0) = 1 \quad \Rightarrow$$

$$(*) \quad f^a(\varepsilon,0) = f^a(0,\varepsilon) = \varepsilon^a$$

Taylor series:

$$f^a(\varepsilon,\tau) = \varepsilon^a + \tau^a + c^{abc}\varepsilon^b\tau^c + \dots$$

Due to $(*)$ it contains no ε^2, τ^2.

representation:

$$D(g(\varepsilon)) = 1 - i\varepsilon^a T^a + \frac{1}{2}\varepsilon^a\varepsilon^b T^{ab} + \dots$$

T^a generator

$T^{ab} = T^{ba}$ symmetric

We must have

$$D(g(\varepsilon))\,D(g(\tau)) = D(g(f(\varepsilon,\tau))) \qquad \Longrightarrow$$

[Note that this is for any representation. For matrix groups this includes the group as well]

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$$(1 - i\varepsilon^a T^a + \tfrac{1}{2}\varepsilon^a\varepsilon^b T^{ab} + \dots)(1 - i\tau^c T^c + \tfrac{1}{2}\tau^c\tau^d T^{cd} + \dots)$$

$$= 1 - i f^a(\varepsilon,\tau) T^a + \tfrac{1}{2} f^a(\varepsilon,\tau) f^b(\varepsilon,\tau) T^{ab} + \dots$$

$$= 1 - i(\varepsilon^a + \tau^a + C^{abc}\varepsilon^b\tau^c) T^a + \tfrac{1}{2}(\varepsilon^a + \tau^a)(\varepsilon^b + \tau^b) T^{ab} \dots$$
$$+ \dots$$

The terms $\sim 1, \varepsilon, \tau, \varepsilon^2, \tau^2$ match automatically.

The terms $\sim \varepsilon^a\tau^b$ give

$$T^{ab} = -T^a T^b + i C^{cab} T^c$$

The C^{abc} are derivatives of the function f^a, and therefore only depend on the group structure and not on the representation. This means that the T^{ab} are determined by the T^a and by the group structure.

$$T^{ab} = T^{ba} \Rightarrow -T^a T^b + i C^{cab} T_c = -T^b T^a + i C^{cba} T_c$$

(*) $$\boxed{[T^a, T^b] = i f^{abc} T^c}$$ Lie-Algebra

with $f^{abc} = C^{cab} - C^{cba}$ structure constants

(*) holds for all representations.

example: $G = SO(3)$ rotations.

J^i = generator of rotations around x^i-axis
= i-th component of angular momentum

Lie algebra: $[J^i, J^k] = i\varepsilon^{ikl} J^l$ ε^{ikl} = Levi-Civita symb.

finite rotation $D(\vec{\theta}) = \exp\{-i\vec{\theta}\cdot\vec{J}\}$

One can show that not only the T^{ab}, but also the coefficients of all higher order terms are determined by the generators, and that in general

$$D(g(\epsilon)) = e^{-i\epsilon^a T^a}$$

Thus representations can be found as follows:

(i) determine the Lie algebra

(ii) find a representation of the Lie algebra

(iii) exponentiate