4.10 Asymptotic behavior

Mass renormalization

$$m_B^2 = m^2 Z_m(\lambda, \varepsilon) \quad , \quad Z_m = 1 + \text{pole terms}$$

$$\mu \frac{d}{d\mu} m_B^2 = 0 \quad \Rightarrow$$

$$\mu \frac{dm^2}{d\mu} Z_m + m^2 \frac{\partial Z_m}{\partial \lambda} \beta(\lambda) = 0 \quad \Rightarrow$$

$$\mu \frac{dm^2}{d\mu} = m^2 \gamma_m(\lambda, \varepsilon) \qquad \text{with}$$

$$\gamma_m(\lambda, \varepsilon) = - Z_m^{-1} \frac{\partial Z_m}{\partial \lambda} \beta(\lambda, \varepsilon)$$

for our example:

$$Z_m(\lambda, \varepsilon) = 1 + \frac{f\lambda}{\varepsilon} + O(\lambda^2) \quad , \quad f = \frac{1}{32\pi^2}$$

$$\gamma_m = -(1 + O(\lambda))^{-1} \left(\frac{f}{\varepsilon} + O(\lambda)\right)\left(-2\varepsilon\lambda + O(\lambda^2)\right)$$

$$= 2f\lambda + O(\lambda^2)$$

Since m is finite for $\varepsilon \to 0$ we may again write

$$\boxed{\mu \frac{dm^2}{d\mu} = m^2 \gamma_m(\lambda)} \qquad \text{with} \quad \gamma_m(\lambda) = \lim_{\varepsilon \to 0} \gamma_m(\lambda, \varepsilon)$$

A solution to this equation is called running mass.
Once the differential eq. for λ_R is solved, the one for m^2 can simply be integrated.

renormalized n-point function:

$$G_R^{(n)}(\mu, \lambda, m, p_1, \ldots) = Z^{-n/2}\, G^{(n)}(\lambda_B, m_B, p_1, \ldots)$$

$Z = Z(\lambda, \varepsilon)$ "Wave function renormalization"

Consider two different scales μ, μ'

$$\zeta(\mu', \mu) := \frac{Z(\lambda', \varepsilon)}{Z(\lambda, \varepsilon)} \qquad \text{Then}$$

$$\frac{G_R^{(n)}(\mu', \ldots)}{G_R^{(n)}(\mu, \ldots)} = \left[\zeta(\mu', \mu)\right]^{-n/2} \quad \text{which is finite.}$$

(*) $$\mu' \frac{d}{d\mu'} \ln \zeta^{1/2}(\mu', \mu) = \mu' \frac{d}{d\mu'} \ln Z^{1/2}(\lambda', \varepsilon) =: \gamma(\lambda', \varepsilon)$$

$$\gamma(\lambda) := \lim_{\varepsilon \to 0} \gamma(\lambda, \varepsilon) \quad \text{is finite}$$

integrate (*):

$$\zeta^{1/2}(\mu', \mu) = \exp\left\{ \int_{\mu}^{\mu'} \frac{d\bar\mu}{\bar\mu}\, \gamma(\lambda(\bar\mu)) \right\}$$

$$\bar\mu \frac{d\lambda}{d\bar\mu} = \beta(\lambda(\bar\mu)) \Rightarrow \frac{d\bar\mu}{\bar\mu} = \frac{d\bar\lambda}{\beta(\bar\lambda)}$$

$$\zeta^{1/2}(\mu', \mu) = \exp\left\{ \int_{\lambda(\mu)}^{\lambda(\mu')} \frac{d\bar\lambda}{\beta(\bar\lambda)}\, \gamma(\bar\lambda) \right\} \Rightarrow$$

(*) $$\boxed{G_R^{(n)}(\mu', \lambda', m', p_1, \ldots, p_n) = \exp\left\{ -n \int_{\lambda(\mu)}^{\lambda(\mu')} d\bar\lambda\, \frac{\gamma(\bar\lambda)}{\beta(\bar\lambda)} \right\} G_R^{(n)}(\mu, \lambda, m, p_1, \ldots, p_n)}$$

where $\lambda' = \lambda(\mu'), \ldots$

(*) describes the change of scale $\mu \to \mu'$ on n-point functions. What's it's use?

If one starts at some scale μ, where perturbation theory for $G_R^{(n)}$ is well behaved (no large logs), and if λ remains small when going from μ to μ',

then (*) gives a reliable result for the scale μ'.

The change of scale $\mu \to \mu'$ is called <u>renormalization group</u> (RG) transformation.

This works even if $\lambda(\mu) \ln \frac{\mu'}{\mu}$ is large, and the usual perturbation theory breaks down.

In this way one can extend the range of validity of perturbation theory (<u>improved</u> perturbation theory)

The crucial question is how the running coupling behaves.

now compute the running coupling for $\mathcal{L}_{int} = -\frac{\lambda}{4!}\varphi^4$:

$$\mu' \frac{d}{d\mu'} \lambda_i (\mu') = 2a\lambda'^2 \quad , \ a = \frac{3}{32\pi^2}$$

$$\frac{d\lambda'}{\lambda'^2} = 2a \frac{d\mu'}{\mu'} \Rightarrow -\left(\frac{1}{\lambda'} - \frac{1}{\lambda}\right) = 2a \ln \frac{\mu'}{\mu}$$

$$\frac{1}{\lambda'} = \frac{1}{\lambda} - 2a \ln \frac{\mu'}{\mu} \ , \quad \lambda' = \frac{1}{\frac{1}{\lambda} - 2a \ln \frac{\mu'}{\mu}}$$

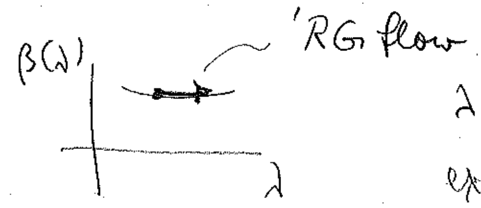
$$\boxed{\lambda(\mu') = \frac{\lambda(\mu)}{1 - \frac{3\lambda(\mu)}{16\pi^2} \ln \frac{\mu'}{\mu}}}$$

$\lambda(\mu')$ grows with μ', and has a pole at large μ', the <u>Landau pole</u>.

N.B. Landau discovered such a pole in quantum electrodynamics, which made him deeply suspicious of QFT.

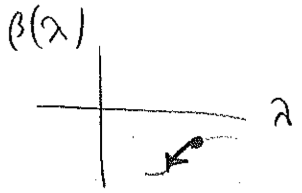
examples: arrow direction indicates growing μ'.

(i)



λ' increases with μ'.
example: $\mathcal{L}_{int} = -\frac{\lambda}{4!}\varphi^4$ in 4 dim.

(ii)

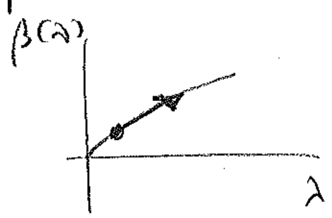


λ' decreases

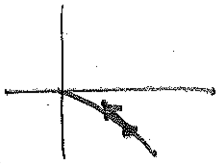
If $\beta(\lambda^*) = 0$, λ^* is called <u>fixed point</u>

example:

(i)



(ii)



$\lambda' \to 0$ for $\mu' \to \infty$

<u>asymptotic freedom</u>

4.7