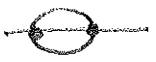
What happens in the 2-loop approximation?

selfenergy

 → mass renormalization

 superficial degree of divergence $= 8 - 6 = 2$

$\frac{\partial}{\partial p^2} \Pi(p^2)$ is UV divergent. $\Rightarrow Z$ is UV divergent.

This divergence cancels in the <u>renormalized Green's functions</u> (see LSZ formula).

$$G_R^{(n)}(p_1, \dots, p_n) = Z^{-n/2}\, G^{(n)}(p_1, \dots, p_n) .$$

A QFT for which the renormalized Green's functions, written in terms of renormalized parameters, are finite to all orders is called (perturbatively) <u>renormalizable</u>.

The choice of renormalized parameters, also called <u>renormalization scheme</u> is not unique.
The one we discussed so far is sometimes called <u>on-shell renormalization</u>.

4.8 Minimal Subtraction

Now we discuss another version, commonly used in perturbative QCD.

mass dimensions: $0 = [S] = \int d^dx\, \mathcal{L} \Rightarrow [\mathcal{L}] = d$

$$[(\partial\varphi)^2] = d \Rightarrow [\varphi^2] = d - 2$$

$$[\lambda_B \varphi^4] = [\lambda_B] + 2d - 4 = d \Rightarrow$$

$$[\lambda_B] = 4 - d$$

define the renormalized coupling such that it is dimensionless also in $d \neq 4$.

At LO:

$$\lambda_B = \mu^{4-d}\lambda = \mu^{2\varepsilon}\lambda$$

with some mass scale μ.

at higher orders in perturbation theory:

$$\lambda_B = \mu^{2\varepsilon}\lambda\left\{1 + a\frac{\lambda}{\varepsilon} + b\frac{\lambda^2}{\varepsilon} + c\frac{\lambda^2}{\varepsilon^2} + d\frac{\lambda^3}{\varepsilon} + \dots\right\}$$

$$m_B^2 = m^2\left\{1 + f\frac{\lambda}{\varepsilon} + g\frac{\lambda^2}{\varepsilon} + \dots\right\}$$

The coefficients $a, b, \dots$ are chosen to cancel the $\frac{1}{\varepsilon^n}$ terms and nothing else.

This is called the <u>Minimal Subtraction scheme</u> or $\overline{MS}$ scheme.

consider again the truncated 4-point function

tree level:

 $= -i\lambda_B = -i\lambda\,\mu^{2\varepsilon}\left(1 + \frac{a\lambda}{\varepsilon} + \dots\right)$

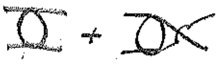
1-loop:

 $= \frac{i}{2}\lambda_B^2\; B(s, m_R)$

$$= \frac{i}{2}\lambda^2\mu^{4\varepsilon}\frac{1}{(4\pi)^2}\left\{\frac{1}{\varepsilon} - \gamma + \ln(4\pi) - \int_0^1 dy\, \ln\left(m_R^2 - y(1-y)s\right)\right\}$$

$$\mu^{2\varepsilon}\frac{1}{\varepsilon} = \frac{1}{\varepsilon}e^{\varepsilon\ln\mu^2} = \frac{1}{\varepsilon} + \ln\mu^2 + O(\varepsilon)$$

$$= \frac{i}{2}\lambda^2\mu^{2\varepsilon}\frac{1}{(4\pi)^2}\left\{\frac{1}{\varepsilon} - \int_0^1 dy\,\ln\left(\frac{m_R^2 - y(1-y)s}{4\pi\mu^2 e^{-\gamma}}\right)\right\}$$

choose a such that in ( $+ \frac{1}{3}$ ) the $\frac{1}{\varepsilon}$ cancels (the other $2/3$ of  are added to ) $\Rightarrow$

$$a = \frac{3}{2(4\pi)^2}$$

Then one can take $\varepsilon \to 0$ and one obtains a finite result:

$$G^{(4)}_{c,\,trunc} = -i\lambda\left\{1 + \frac{\lambda_R}{2(4\pi)^2}\int_0^1 dy\left[\ln\frac{m_R^2 - y(1-y)s}{4\pi\mu^2 e^{-\gamma}} + (s\to t) + (s\to u)\right]\right\}$$

again the logarithm of dimensionful quantities is gone.

[In the so-called modified minimal subtraction ($\overline{MS}$) scheme not only $\frac{1}{\varepsilon}$, but also $-\gamma + \ln 4\pi$ is removed]

mass renormalization

inverse full propagator

$i\Big[$  $\Big]^{-1} = p^2 - m_B^2 - \Pi(p^2)$

$$= p^2 - m^2\left(1 + \frac{f\lambda}{\varepsilon} + \dots\right) - \Pi(p^2)$$

from  we had

$$\Pi(p^2) = + \frac{\lambda_B}{2} A(m_B^2) + O(\lambda_B^2)$$

$$\frac{\lambda_B}{2} A = -\frac{m^2}{(4\pi)^2}\frac{\lambda}{2}\left(\frac{1}{\varepsilon} + \ln\mu^2 + 1 - \gamma - \ln\frac{m^2}{4\pi}\right)$$

Choose

$$f = \frac{1}{2(4\pi)^2} \quad \Rightarrow$$

$i\Big[$  $\Big]^{-1} = p^2 - m^2\left(1 - \frac{\lambda}{2(4\pi)^2}\left[1 - \ln\frac{m^2}{4\pi\mu^2 e^{\gamma}}\right]\right) + O(\lambda^2)$

note that the MS mass differs from the pole mass!

4.9 The renormalization group

[Brown 5.3 – 5.4] [Collins 7.3]

We have seen that in the $O(\lambda_R^2)$ corrections to M logarithms like $\ln\frac{s}{m^2}$ or $\ln\frac{s}{\mu^2}$ appear.

One expects $\lambda\left(\lambda \ln\frac{s}{\mu^2}\right)^n$ corrections at higher orders.

At high energies these logs can become large, and $\lambda \cdot \ln\frac{s}{\mu^2}$ can be large even if λ is small, so that perturbation theory does not work anymore.

One can improve perturbation theory by summing these so-called leading-log corrections.

Clearly, $\ln\frac{s}{\mu^2}$ would not be large when $\mu^2 \simeq s$. Are we free to choose μ?

Remember that we started out with λ_B, m_B which were independent of μ. Thus, when we change μ, λ and m will have to change.

Our discussion will be rather general. [It applies not only to scalar φ^4 theory.] We use the MS scheme.

The bare coupling can be written as

$$\lambda_B = \mu^{2\varepsilon} F(\lambda, \varepsilon) \quad , \quad F = \lambda + \text{pole terms}$$

example: for $\mathcal{L}_{int} = -\frac{\lambda}{4!}\varphi^4$

$$F(\lambda, \varepsilon) = \lambda + \frac{a\lambda^2}{\varepsilon} + O(\lambda^2) \quad , \quad a = \frac{3}{32\pi^2}$$

$$\mu \frac{d}{d\mu}\lambda_B = 0 \quad , \quad \mu\frac{d}{d\mu} = \frac{d}{d\ln\mu} \, , \quad \mu^{2\varepsilon} = e^{2\varepsilon\ln\mu} \quad \Rightarrow$$

$$2\varepsilon F + \frac{\partial F}{\partial\lambda}\mu\frac{d\lambda}{d\mu} = 0 \quad \Rightarrow$$

The change of λ with μ is determined by the differential equation

$$\mu\frac{d\lambda}{d\mu} = \beta(\lambda, \varepsilon)$$

with

$$\beta(\lambda, \varepsilon) = -\left(\frac{\partial F}{\partial\lambda}\right)^{-1} 2\varepsilon F$$

for our example:

$$\beta = -\left(1 + 2\frac{a\lambda}{\varepsilon}\right)^{-1} 2\varepsilon\left(\lambda + \frac{a\lambda^2}{\varepsilon}\right) + O(\lambda_R^3)$$

$$= -2\varepsilon\lambda \underbrace{\left(1 - 2\frac{a\lambda}{\varepsilon}\right)\left(1 + \frac{a\lambda}{\varepsilon}\right)}_{= 1 - a\lambda/\varepsilon} + O(\lambda_R^3)$$

$$= -2\varepsilon\lambda + 2a\lambda^2 + O(\lambda^3)$$

since λ is finite for $\varepsilon \to 0$, $\beta(\lambda) := \lim_{\varepsilon\to 0}\beta(\lambda, \varepsilon)$ is finite.
The μ dependence of λ_R is determined by

(*) $$\boxed{\mu\frac{d\lambda}{d\mu} = \beta(\lambda)}$$ renormalization group equation (RGE)

β is called <u>β - function</u>

A solution to (*) is called <u>running coupling</u>

μ is called <u>renormalization scale</u>