4.6 4-point function

Amputated n-point functions $G^{(n)}_{amp}$ are defined as $G^{(n)}$ without external propagators.

$G^{(n)}_{amp}$ is what enters the S-matrix.
We consider $G^{(4)}_{c,amp}(-p_1,-p_2,p_3,p_4)$, which enters $\langle \vec{p}_3\,\vec{p}_4|S-1|\vec{p}_1\,\vec{p}_2\rangle$, at $\mathcal{O}(\lambda^2)$. There are 3 diagrams.
One of them is  $= \frac{1}{2}(-i\lambda)^2 \underbrace{\int \frac{d^dk}{(2\pi)^d}\Delta_F(k)\Delta_F(k+p_1+p_2)}_{=:-iB}$

B is Lorentz invariant. Thus it can only depend on the invariant $p^2=(p_1+p_2)^2$ and on m:

$$-iB(p^2,m^2)=\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2+i\varepsilon}\,\frac{i}{(k+p)^2-m^2+i\varepsilon}$$

combine the two denominators using

$$\frac{1}{A_1\cdots A_n}=\int_0^1 dx_1\ldots dx_n\,\delta\Big(1-\sum_{i=1}^n x_i\Big)\frac{(n-1)!}{(x_1A_1+\cdots+x_nA_n)^n}$$

The x_i are called Feynman parameters

$$-iB(p^2,m)=-\int_0^1 dx\,dy\,\delta(1-x-y)$$
$$\cdot\int\frac{d^dk}{(2\pi)^d}\Big[x(k^2-m^2+i\varepsilon)+y(\underbrace{(k+p)^2}_{=k^2+2k\cdot p+p^2}-m^2+i\varepsilon)\Big]^{-2}$$

$$=-\int_0^1 dy\int\frac{d^dk}{(2\pi)^d}\left[k^2+2y\,k\cdot p+yp^2-m^2+i\varepsilon\right]^{-2}$$

complete the square $\Rightarrow$

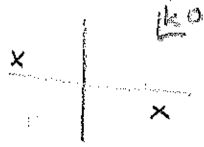
$$iB=\int_0^1 dy\int\frac{d^dk}{(2\pi)^d}\left[(k+yp)^2+y(1-y)p^2-m^2+i\varepsilon\right]^{-2}$$

substitute $k \to k - yp$

(*) $$iB(p^2;m) = \int_0^1 dy \int \frac{d^dk}{(2\pi)^d} \left[k^2 + y(1-y)p^2 - m^2 + i\varepsilon\right]^{-2}$$

Now try to Wick-rotate.

For the 2-point function we had $\propto \int d^dk\,(k^2 - m^2 + i\varepsilon)^{-1}$; the poles were always close to the real k^0-axis.

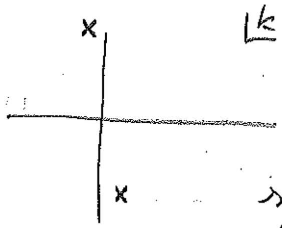


In (*) we have poles at $k_0^2 = \vec{k}^2 + m^2 - y(1-y)p^2 - i\varepsilon$.

For the S-matrix element $\langle \vec{p}_3 \vec{p}_4 | S - 1 | \vec{p}_1 \vec{p}_2 \rangle$ we need

$$p^2 = (p_1 + p_2)^2 = s > 0, \quad \sqrt{s} = \text{center-of-mass energy}$$

For $y(1-y)p^2 > \vec{k}^2 + m^2$ the poles are at



since the poles do not cross the real or imaginary axis.

One can then still Wick rotate, but one has to keep $i\varepsilon$.

$$iB(p^2,m) = i\int_0^1 dy \int \frac{d^d\underline{k}}{(2\pi)^d} \left[-\underline{k}^2 + y(1-y)p^2 - m^2 + i\varepsilon\right]^{-2}$$

We had found

$$\int \frac{d^d\underline{k}}{(2\pi)^d} \frac{1}{(\underline{k}^2 + M^2)^n} = \frac{M^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)}$$

Now we have $n = 2$ and

$$M^2 = m^2 - y(1-y)p^2 - i\varepsilon$$

$$B(p^2, m) = \int_0^1 dy \, \frac{1}{(4\pi)^{d/2}} \left[m^2 - y(1-y)p^2 - i\varepsilon \right]^{\frac{d}{2}-2} \cdot \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}$$

$$\Gamma(2 - \tfrac{d}{2}) = \Gamma(2 - (4-2\varepsilon)/2) = \Gamma(\varepsilon) = \tfrac{1}{\varepsilon}\Gamma(1+\varepsilon)$$
$$= \tfrac{1}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon)$$

$$\Gamma(2) = 1$$

$$(M^2)^{d/2-2} = (M^2)^{-\varepsilon} = e^{-\varepsilon \ln M^2} = 1 - \varepsilon \ln M^2 + \mathcal{O}(\varepsilon^2)$$

$$(4\pi)^{-d/2} = (4\pi)^{-2+\varepsilon} = (4\pi)^{-2}\left(1 + \varepsilon \ln(4\pi)\right) + \mathcal{O}(\varepsilon^2)$$

$\Rightarrow$

$$B(p^2, m) = \frac{1}{(4\pi)^2} \left\{ \frac{1}{\varepsilon} - \gamma + \ln(4\pi) - \int_0^1 dy \, \ln\left(m^2 - y(1-y)p^2 - i\varepsilon\right) \right\}$$

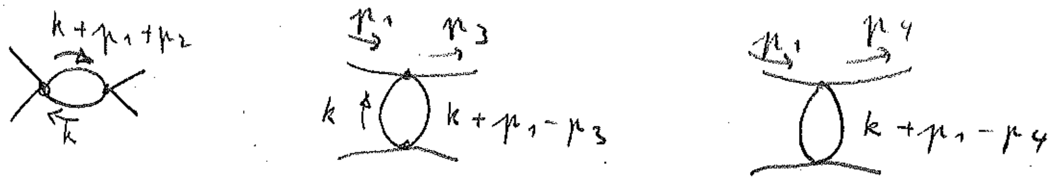
N.B. $y(1-y) < \frac{1}{4} \Rightarrow$ for $p^2 < 4m^2$ one can put $\varepsilon \to 0$ and the Feynman-parameter integral is real. For $p^2 > 4m^2$ it has an imaginary part. This is related to the optical theorem which states that

Im (forward scattering amplitude) $\propto \sigma_{tot}$

at $\mathcal{O}(\lambda^2)$: $\sigma_{tot} \propto \Big|$  $\Big|^2$

for on shell p_1, p_2: $(p_1 + p_2)^2 > (2m)^2 = 4m^2$

all diagrams contributing to $G^{(4)}_{c,\,amp}(-p_1,-p_2,p_3,p_4)$:



$s=(p_1+p_2)^2 \qquad t=(p_1-p_3)^2 \qquad u=(p_1-p_4)^2$

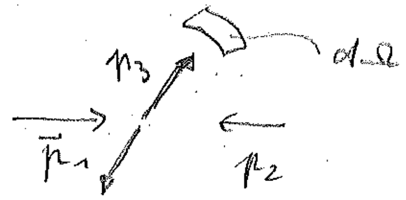
$$G^{(4)}_{c,\,trunc}(-p_1,-p_2,p_3,p_4) =$$

$$= -\frac{i}{2}(-i\lambda)^2\left\{ B(s,m) + B(t,m) + B(u,m)\right\}$$

So we found that the 1-loop selfenergy and 1-loop 4-point functions are ultraviolet divergent. The divergent part is momentum-independent.

4.7 Renormalization of the coupling

The strength of the interaction between particles can be determined in a scattering experiment, e.g. by measuring the cross section $\frac{d\sigma}{d\Omega}$ for $2 \to 2$ scattering for certain momenta.



center of mass system

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{16\pi^2 s} \qquad \mathcal{M} \text{ invariant matrix element}$$

$$\langle \vec{p}_3 \vec{p}_4 | S - 1 | \vec{p}_1 \vec{p}_2 \rangle = (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)\, i\mathcal{M}$$

We had
$$i\mathcal{M} = G^{(4)}_{c,\,amp}(-p_1, -p_2, p_3, p_4)$$

and
$$\mathcal{M} = \mathcal{M}(s,t,u)$$

At leading order:

$\mathcal{M} = -\lambda_B$, momentum-independent.

At higher order $\mathcal{M}$ depends on momenta.

Consider small momenta $|\vec{p}_i| \ll m \Rightarrow s \simeq 4m^2 \gg |t|, |u|$

Define the renormalized coupling by

$$\lambda = -\mathcal{M}(4m^2, 0, 0)$$

note that this has no imaginary part.

We had

$$\mathcal{M}(s,t,u) = -\lambda_B + \frac{\lambda_B^2}{2}\{B(s,m_B) + B(t,m_B) + B(u,m_B)\} + O(\lambda_B^3) \implies$$

$$\lambda_B = \lambda + \frac{\lambda^2}{2}\{B(4m_R^2, m) + 2B(0,m)\} + O(\lambda^3)$$

Insert this into $\mathcal{M}$.

$$\mathcal{M}(s,t,u) = -\lambda + \frac{\lambda^2}{2}\{B(s,m^2) + B(t,m^2) + B(u,m^2) - B(4m^2, m^2) - 2B(0,m^2)\}$$

Since the divergent part of B is momentum independent,

$$B(p^2,m) = \frac{1}{(4\pi)^2}\left\{\frac{1}{\varepsilon} - \gamma + \ln 4\pi - \int_0^1 dy \ln\left(m^2 - y(1-y)p^2 - i\varepsilon\right)\right\}$$

the divergences cancel, and one obtains a finite result for $d \to 4$ $(\Leftrightarrow \varepsilon \to 0)$

$$\mathcal{M}(s,t,u) = -\lambda\left\{1 - \frac{\lambda}{32\pi^2}\int_0^1 dy\left[\ln\left(\frac{m^2 - y(1-y)s - i\varepsilon}{m^2(1-4y(1-y))}\right) + \ln\left(\frac{m^2 - y(1-y)t}{m^2}\right) + \ln\left(\frac{m^2 - y(1-y)u}{m^2}\right)\right]\right\},$$

N.B. The logarithms of dimensionful quantities have disappeared as well.

To summarize, we have found that in the 1-loop approximation all divergences cancel, if Green's functions are written in terms of the renormalized quantities m^2, λ^2.