4.4 Dimensional regularization

[Peskin, Schröder 10.2]

- preserves Lorentz invariance
- only defined in perturbation theory

idea: consider the theory in $d<4$ dimensions

$$A(m^2) \to \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+m^2} = \frac{\Omega_d}{(2\pi)^d} \int_0^\infty d\rho\, \rho^{d-1} \frac{1}{\rho^2+m^2}$$

Compute

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(\underline{k}^2+m^2)^n} = \frac{\Omega_d}{(2\pi)^d} \int_0^\infty d\rho \frac{\rho^{d-1}}{(\rho^2+m^2)^n}$$

[later we will also need $n>1$].

Subst: $x := \frac{m^2}{\rho^2+m^2}$, $\rho^2+m^2 = \frac{m^2}{x}$, $\rho^2 = m^2\left(\frac{1}{x}-1\right) = m^2 \frac{1-x}{x}$

$$\rho = m(1-x)^{1/2} x^{-1/2}$$

$$d\rho = m(1-x)^{-1/2} x^{-3/2} \left[+ \tfrac{1}{2}(-1) x - \tfrac{1}{2}(1-x) \right] dx$$

$$= -\tfrac{1}{2} m (1-x)^{-1/2} x^{-3/2} dx$$

$$\int_0^\infty d\rho \frac{\rho^{d-1}}{(\rho^2+m^2)^n} = +\tfrac{1}{2} m^{d-2n} \int_0^1 dx\, (1-x)^{-1/2+\frac{d-1}{2}} x^{-3/2-\frac{d-1}{2}+n}$$

$$= \frac{m^{d-2n}}{2} \int_0^1 dx\, (1-x)^{d/2-1} x^{-d/2-1+n}$$

$$\int_0^1 dx\, (1-x)^{\alpha-1} x^{\beta-1} = \underset{\text{Beta function}}{B(\alpha,\beta)} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \Rightarrow$$

Euler Γ-function

$$\int_0^\infty d\rho \frac{\rho^{d-1}}{(\rho^2+m^2)^n} = \frac{m^{d-2n}}{2} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}$$

Compute Ω_d:

$$(\sqrt{\pi})^d = \left(\int dx\, e^{-x^2}\right)^d = \int d^d x \exp\left(-\sum_{i=1}^d x_i^2\right) = \int d\Omega_d \int_0^\infty dx\, x^{d-1} e^{-x^2}$$

$$= \Omega_d \int_0^\infty \tfrac{1}{2} y^{-1/2} dy\, y^{\frac{d-1}{2}} e^{-y} = \frac{\Omega_d}{2} \int_0^\infty dy\, y^{d/2-1} e^{-y} = \frac{\Omega_d}{2} \Gamma\left(\frac{d}{2}\right)$$

$y = +x^2$

$$\Rightarrow \boxed{\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}}$$

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QFT

$$\int \frac{d^d \underline{k}}{(2\pi)^d} \frac{1}{(\underline{k}^2+m^2)^n} = \frac{\pi^{d/2}}{(4\pi^2)^{d/2}}\, m^{d-2n} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}$$

$$= \frac{m^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}$$

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4.5 Mass renormalization

[Peskin, Schröder 10.2]

The parameters in the Lagrangian are called bare (nackt). To clearly distinguish them from the renormalized parameters (to be defined below) write them with a subscript B:

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{m_B^2}{2}\varphi^2 - \frac{\lambda_B}{4!}\varphi^4$$

In order to make predictions, one has to express the bare parameters in terms of measured quantities. The choice of these quantities is not unique.

One possible choice consists of the physical particle mass m, and λ, defined as i times the $2 \to 2$ particle scattering amplitude for a certain choice for s, t, u.

m and λ are called renormalized mass and coupling.

At lowest order: $m_B = m, \quad \lambda_B = \lambda$.

Now we include the $O(\lambda_B)$ correction to m.

$G^{(2)}(p) = \dfrac{i}{p^2 - m_B^2 - \Pi(p^2) + i\varepsilon}$ has a pole at $p^2 = m^2$ $\Rightarrow$

$$m^2 = m_B^2 + \Pi(m^2)$$

We had $\quad -i\,\Pi(p^2) =$  $+\, O(\lambda_B^2)$

$$= + i\frac{\lambda_B}{2} A(m_B^2) + O(\lambda_B^2)$$

$$A(m^2) = \int \frac{d^dk}{(2\pi)^d} \Delta_F(k)$$

We found

$$A(m^2) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+m^2}$$

and $\displaystyle\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+m^2)^n} = \frac{m^{d-2n}}{(4\pi)^{d/2}} \frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}$

Here we need $n=1$.

define ε such that $d = 4-2\varepsilon$

$$\Gamma(1-\tfrac{d}{2}) = \Gamma(-1+\varepsilon)$$

$$\Gamma(1+x) = x\,\Gamma(x) \implies \Gamma(1+\varepsilon) = \varepsilon\,\Gamma(\varepsilon) = \varepsilon(\varepsilon-1)\Gamma(-1+\varepsilon)$$

$$\Gamma(-1+\varepsilon) = -\frac{1}{\varepsilon}\frac{1}{1-\varepsilon}\Gamma(1+\varepsilon)$$

poles at $\varepsilon = 0, 1$ corresponding to $d = 4, 2$

$$A(m^2) = \frac{m^{2-2\varepsilon}}{(4\pi)^{2-\varepsilon}}(-1)\frac{1}{\varepsilon}\frac{1}{1-\varepsilon}\Gamma(1+\varepsilon)$$

$$= -\frac{m^2}{(4\pi)^2}\left(\frac{m^2}{4\pi}\right)^{-\varepsilon}\frac{1}{\varepsilon}\frac{1}{1-\varepsilon}\Gamma(1+\varepsilon)$$

analytically continue ε and expand around $\varepsilon = 0$.

$\Gamma(1+\varepsilon) = 1 - \gamma\varepsilon + O(\varepsilon^2)$ $\gamma = 0.577$ Euler Mascheroni const.

$$\left(\frac{m^2}{4\pi}\right)^{-\varepsilon} = e^{-\varepsilon \ln\frac{m^2}{4\pi}} = 1 - \varepsilon \ln\frac{m^2}{4\pi}$$

$$A(m^2) = -\frac{m^2}{(4\pi)^2}\left(\frac{1}{\varepsilon} + 1 - \gamma - \ln\frac{m^2}{4\pi}\right) + O(\varepsilon)$$

compare with (i): $\Omega_4 = 2\pi^2$ $(\Gamma(2)=1)$

$$A(m^2) = \frac{2\pi^2}{(2\pi)^4}\frac{1}{2}\left(\Lambda^2 - m^2\ln\frac{\Lambda^2}{m^2}\right) = \frac{1}{(4\pi)^2}\left(\Lambda^2 - m^2\ln\frac{\Lambda^2}{m^2}\right)$$

quadratic, logarithmic divergences correspond to poles at $\varepsilon = 1, 0$

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$$m^2 = m_B^2 + \Pi(m^2) = m_B^2 + \frac{\lambda_B}{2} A(m_B^2) + O(\lambda_B^2)$$

solve for m_B $\Rightarrow$

$$m_B^2 = m^2 - \frac{\lambda_B}{2} A(m_B^2) + O(\lambda_B^2)$$

$$= m^2 - \frac{\lambda}{2} A(m^2) + O(\lambda^2)$$

By writing $G^{(2)}$ in terms of m, λ the divergence disappear:

$$G^{(2)}(p) = \frac{i}{p^2 - m^2} + O(\lambda^2)$$

N.B. At this order $Z = 1$ (Z = wave function renormalization)

because Π is p-independent here.