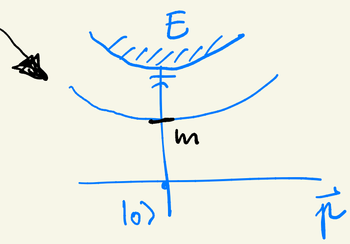
4 Renormalization

4.1 Full propagator

We have seen that the exact (or full) 2-point function (or full propagator) has a pole on the 1-particle mass shell,

$$G^{(2)}(p) = \frac{i Z}{p^2 - m^2 + i\varepsilon} + \text{non-singular terms}$$

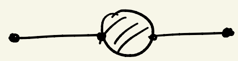


m the physical particle mass. It is also called a renormalized mass, in contrast to the bare unrenormalized mass m_B.

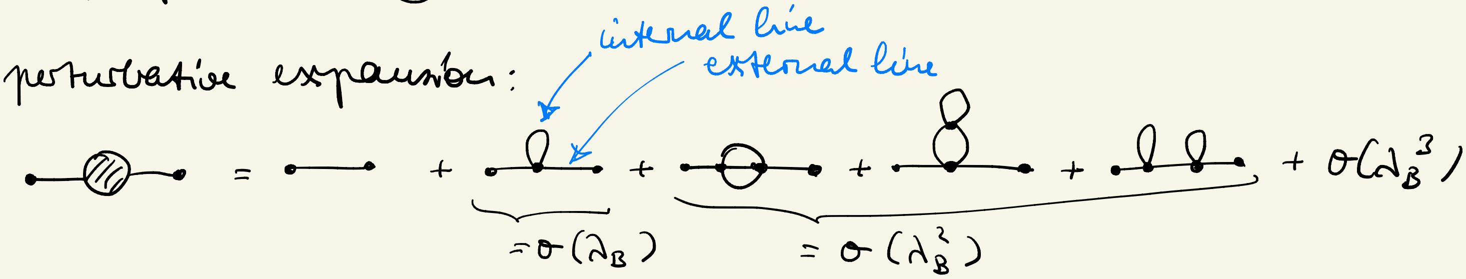
Again consider $\quad \mathcal{L} = \frac{1}{2}\partial\varphi^2 - \frac{m_B^2}{2}\varphi^2 - \frac{\lambda_B}{4!}\varphi^4 .$

One can compute m and Z from $\mathcal{L}$ and write them in terms of m_B and λ_B.

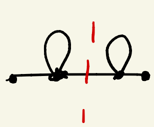
Draw the full propagator as follows:

$G^{(2)}(p) =$ 

perturbative expansion:

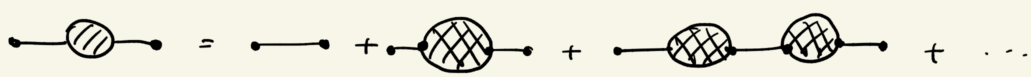


A diagram which can be cut into two by cutting one internal line is called 1-particle reducible

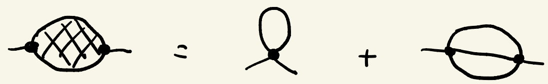
example:  is 1-particle reducible

A diagram which is not 1-particle reducible is called
<u>1-particle irreducible</u> or <u>1PI</u>

Define  as the sum of 1PI diagrams for 2 external lines.



Perturbative expansion of  :

 $+ \mathcal{O}(\lambda_B^3)$

<u>Define</u> the function $\Pi(p^2)$ through

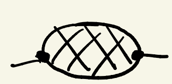
$$G^{(2)}(p) = \frac{i}{p^2 - m_B^2 + i\varepsilon - \Pi(p^2)}$$

Π is called <u>selfenergy</u> and is Lorentz-invariant.

$$G^{(2)}(p) = \frac{i}{p^2 - m_B^2 + i\varepsilon}\left[1 - \frac{\Pi}{p^2 - m_B^2 + i\varepsilon}\right]^{-1} \qquad \text{(Taylor exp.)}$$

$$= \frac{i}{p^2 - m_B^2 + i\varepsilon} + \frac{i}{p^2 - m_B^2 + i\varepsilon}(-i\Pi)\frac{1}{p^2 - m_B^2 + i\varepsilon} + \cdots$$

Thus we have

$-i\Pi =$ 

Taylor-expansion around 1-particle mass shell:

$$\Pi(p^2) = \Pi(m^2) + \Pi'(m^2)\,(p^2 - m^2) + \dots \quad \text{with}$$

$$\Pi'(m^2) := \left.\frac{d\Pi}{dp^2}\right|_{p = m^2} \qquad \Rightarrow$$

$$G^{(2)}(p) \simeq \frac{i}{p^2 - m_B^2 + i\varepsilon - \Pi(m^2) - \Pi'(m^2)(p^2 - m^2)} \simeq \frac{Z}{p^2 - m^2} \Rightarrow$$

$$\boxed{m^2 = m_B^2 + \Pi(m^2)} \qquad m: \text{particle mass}$$

m defined this way is also called <u>pole mass</u>.

Then

$$\frac{i}{(p^2 - m^2)(1 - \Pi'(m^2))} = \frac{Z}{p^2 - m^2} \qquad \Rightarrow$$

$$\boxed{Z^{-1} = 1 - \Pi'(m^2)}$$

4.2 Counting divergences

[Ryder 9.1, 9.2]

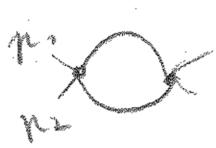
We have seen that in perturbation theory integrals over internal, loop momenta can be divergent.

examples: (i)  $= -i\frac{\lambda_B}{2} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_B^2 + i\varepsilon}$ quadratically divergent

this contributes to the selfenergy

 $= -i\frac{\lambda_B}{2} \langle \varphi^2(x) \rangle_0 = -i\frac{\lambda_B}{2} \langle \varphi^2(0) \rangle_0$

The fact that this is divergent also means that the operator φ^2 is not (yet) well defined.

(ii) 

$= (-i\lambda_B)^2 \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m_B^2 + i\varepsilon} \frac{i}{(p - p_1 - p_2)^2 - m_B^2 + i\varepsilon}$

logarithmically divergent

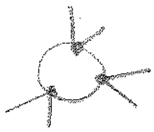
note that this is proportional to the FT of $\langle \varphi^2(x) \varphi^2(0) \rangle_0$

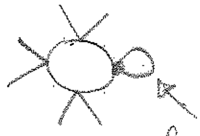
Define the superficial degree of divergence D as

$D =$ # powers of momentum in the numerator
$-$ # powers " " denominator

naively: divergent when $D \geq 0$, convergent when $D < 0$,

 $D = 2$  $D = 0$

 $D = -2$ convergent

however:  $D = 2\cdot 4 - 5\cdot 2 = -2$

but there is a divergent subdiagram

Determine D by dimensional analysis:

$[\varphi(x)] = 1$ (mass dimension) $\Rightarrow$

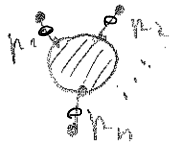
$[G^{(n)}(x_1,\dots)] = n$ FT: $\left[\int d^4x\right] = -4 \Rightarrow$

$[\tilde{G}^{(n)}(p_1,\dots)] = n - 4n = -3n$

Define $G^{(n)}(p_1,\dots)$ through

$$\tilde{G}^{(n)}(p_1,\dots,p_n) = (2\pi)^4 \underbrace{\delta(p_1+\dots)}_{\text{dim.} = -4} G^{(n)}(p_1,\dots)$$

$[G^{(n)}(p_1,\dots)] = -3n + 4$

$G^{(n)}(p_1,\dots) =$ 

Define the <u>amputated</u> <u>Green's function</u> $G^{(n)}_{amp}$ as $G^{(n)}$ without the external propagators:

$G^{(n)}_{amp}(p_1,\dots) =$ 

$[G^{(n)}_{amp}(p_1,\dots)] = -3n + 4 + 2n = 4 - n$; $[\lambda] = 0 \Rightarrow$

$\boxed{D = 4 - n}$ to <u>any</u> order in perturbation theory! Only 2- and 4-point functions are superficially divergent.

this indicates that only a small number of divergences has to be "removed".

4.3 Regularization

We have to modify, regularize, our theory to give a meaning to the divergent integrals.

Again consider

 $= -i\lambda_B \frac{1}{2} A(m_B^2)$

$k_0^2 = (\mathrm{Re}\,k_0)^2 + 2i\,\mathrm{Re}\,k_0\,\mathrm{Im}\,k_0 - (\mathrm{Im}\,k_0)^2$

with $A(m^2) := \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon}$

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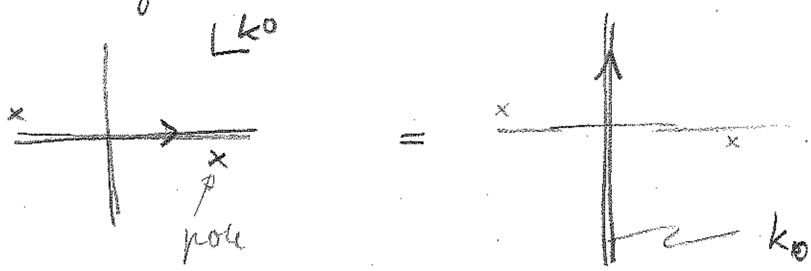

$\int dk_0 \to i \int d\underline{k}_0$ Wick rotation

$$A(m^2) = \int \frac{d^4\underline{k}}{(2\pi)^4} \frac{1}{\underline{k}^2 + m^2}$$ with $\underline{k}^2 = \underline{k}_0^2 + \vec{k}^2$ "Euclidean"

4-dimensional spherical coordinates ($\vec{\underline{k}} = \vec{k}$)

$$\int d^4\underline{k} = \int d\Omega_4 \int_0^\infty d\rho\, \rho^3$$

$$A = \frac{\Omega_4}{(2\pi)^4} \int_0^\infty d\rho\, \rho^3 \frac{1}{\rho^2 + m^2}$$ diverges at large ρ

$\Omega_4 = \int d\Omega_4$ = volume of 3-dim unit sphere possible regularizations:

(i) cut-off $\int_0^\infty dp \to \int_0^\Lambda dp$

Λ <u>momentum cutoff</u> <u>UV - cutoff</u>

$$A = \frac{\Omega_4}{(2\pi)^4} \underbrace{\int_0^\Lambda dp\, p^3 \frac{1}{p^2+m^2}} = \frac{\Omega_4}{(2\pi)^4} \frac{1}{2}\left(\Lambda^2 - m^2 \ln \frac{\Lambda^2+m^2}{m^2}\right)$$

$$= \frac{1}{2}\int_0^{\Lambda^2} dp^2 \frac{p^2+m^2-m^2}{p^2+m^2}$$

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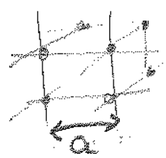
quadratic and logarithmic divergences

(ii) space-time $\to$ discrete cubic lattice



<u>lattice regularization</u>

a : lattice spacing

momentum $k \in$ Brillouin zone, $-\frac{\pi}{a} \leq k_\mu \leq \frac{\pi}{a}$

- well defined beyond perturbation theory
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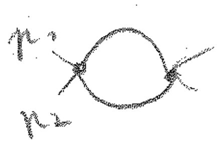
(ii)  $= (-i\lambda_B)^2\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m_B^2+i\varepsilon}\frac{i}{(p-p_1-p_2)^2-m_B^2+i\varepsilon}$

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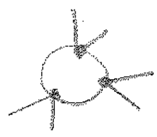
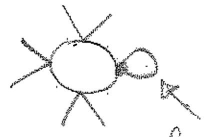
 $D = -2$ convergent

however:  $D = 2\cdot 4 - 5\cdot 2 = -2$

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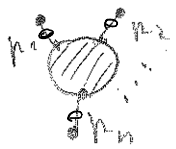
$G^{(n)}(p_1,\dots) =$ 

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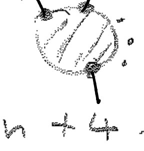
$G^{(n)}_{amp}(p_1,\dots) =$ 

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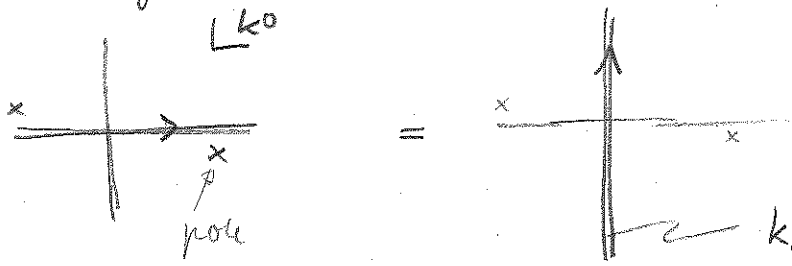

$k_0 = i\underline{k}_0, \quad \underline{k}_0 \in \mathbb{R}$

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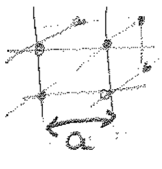

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