3.8 4-point function

$$G^{(4)}(x_1,\dots,x_4) = N\int \mathcal{D}\varphi\, e^{i(S_0+S_{int})}\,\varphi(x_1)\dots\varphi(x_4)$$
$$= G_0^{(4)} + G_1^{(4)} + \mathcal{O}(\lambda_B^2)$$

for $S_{int} = \int d^4x\,\mathcal{L}_{int}$, $\mathcal{L}_{int} = -\frac{\lambda_B}{4!}\varphi^4$

We had $G_0^{(4)} =$  → no scattering

$$G_1^{(4)} = \langle\varphi(x_1)\dots\varphi(x_4)\, iS_{int}\rangle_0\Big|_{\text{no vacuum subdiagrams}}$$

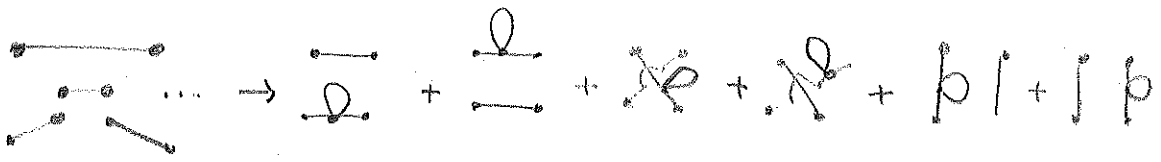
We have to contract

three types of contractions:

(i) all x_i are contracted among themselves, →

has vacuum subdiagram, does not contribute to $G^{(4)}$

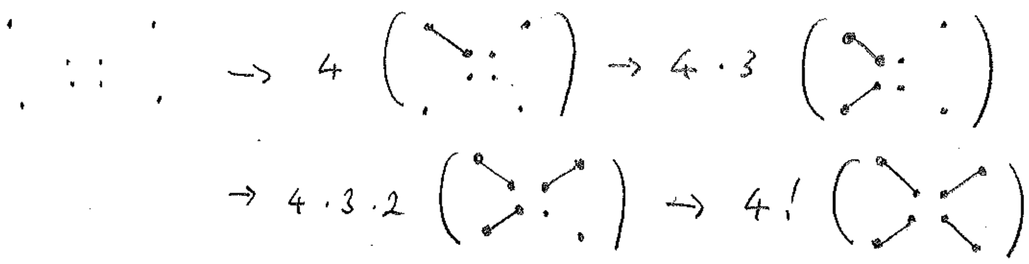
(ii) two x_i are contracted among themselves



disconnected

These can be obtained from $G_0^{(4)}$ by replacing one $G_0^{(2)}$ by $G_1^{(2)}$.

(iii) contract all x_i with x



→ Feynman diagram  connected

$$\langle \varphi(x_1) \cdots \varphi(x_4)\, \varphi^4(x) \rangle_{0,c} = 4!\, \Delta_F(x_1 - x) \cdots \Delta_F(x_4 - x)$$

$$G^{(4)}_{1,c}(x_1, \ldots, x_4) = -i\lambda \int d^4x\, \Delta_F(x_1 - x) \cdots \Delta_F(x_4 - x)$$

Fourier transform:

$$\tilde{G}^{(4)}_{1,c}(p_1,\dots,p_4) := \int d^4x_1 \dots d^4x_4\, e^{i(p_1x_1+\dots+p_4x_4)} G^{(4)}_{1,c}(x_1,\dots)$$

$$= -i\lambda_B \int d^4x\, d^4x_1 \cdots d^4x_4\, e^{i(p_1+\dots+p_4)x}\, e^{i(p_1x_1+\dots+p_4x_4)} \qquad (*)$$

$$\cdot \Delta_F(x_1) \cdots \Delta_F(x_4)$$

$$= -i\lambda_B (2\pi)^4\, \delta^4(p_1+\dots+p_4)\, \Delta_F(p_1) \cdots \Delta_F(p_4)$$

Here $\Delta_F(p) = \frac{i}{p^2 - m_B^2 + i\varepsilon}$. We have $m_B^2 = m^2 + \mathcal{O}(\lambda_B)$.
particle mass

$\Rightarrow$ At the order under consideration, $\tilde{G}^{(4)}$ has poles at $p_i^2 = m^2$.

S-matrix element for $2 \to 2$ scattering:

$$\langle \vec{p}_3 \vec{p}_4 | S - 1 | \vec{p}_1 \vec{p}_2 \rangle = Z^{-2} \left[\prod_a \lim_{p_a^0 \to E_{p_a}} (p_a^2 - m^2) \right] \tilde{G}^{(4)}(-p_1, -p_2, p_3, p_4)$$

At leading order: $Z = 1$. $\Rightarrow$

$$\langle \vec{p}_3 \vec{p}_4 | S - 1 | \vec{p}_1 \vec{p}_2 \rangle = -i\lambda_B (2\pi)^4\, \delta^4(p_1 + p_2 - p_3 - p_4) + \mathcal{O}(\lambda_B^2)$$

The disconnected part of $G^{(4)}$ does not contribute to the S-matrix element.

3.9 Feynman rules

So far we have computed some n-point functions by hand, expanding $e^{iS_{int}}$ and applying Wick's theorem. One can derive a set of rules, the Feynman rules, which systematically yield expressions for any n-point function to any order in perturbation theory.
These can even be applied automatically on a computer.

The remaining difficulty is then to perform the resulting integrations.
[Another problem is that the number of diagrams grows very rapidly with increasing order of perturbation theory]

I will simply state the rules for

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

and we will see how these reproduce our previous results, and also how to obtain new ones.

To obtain $G^{(n)}$ at q-th order in λ,

- draw all topologically distinct diagrams with n 'external' lines and q internal points, vertices, at which 4 lines end.

example: $n = 4,\ q = 1$  ← vertex

⇒ Each vertex gives a factor $(-i\lambda)$.

– Some diagrams have certain symmetries, which gives an additional factor, the <u>symmetry factor S</u>.
The rule for S is complicated, we will obtain it by 'hand'!

example: $n = 2,\ q = 1$ 

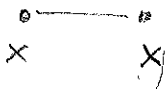
applying Wick's theorem there are only $4 \cdot 3$ contractions, so that the $1/4!$ is not cancelled completely → $S = \frac{1}{2}$

$G^{(n)}$ = sum of all Feynman diagrams without vacuum subdiagrams.

<u>Coordinate space rules</u>

$$G^{(n)}(x_1, \dots, x_n) = \langle 0 | T \varphi(x_1) \cdots \varphi(x_n) | 0 \rangle$$

– The end of each line corresponds to a space-time point.

 $= \Delta_F(x - x')$

for a vertex at x there is an integral $\int d^4x$

 $= -i\lambda \int d^4x$

example:

$n=2,\ q=1$:  $= -i\frac{\lambda}{2} \int d^4x\ \Delta_F(x_1 - x)\, \Delta_F(0)\, \Delta_F(x - x_2)$

Momentum space rules

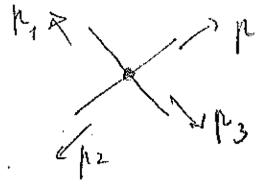
$$(*) \quad \widetilde{G}^{(n)}(p_1, \dots, p_n) = \int d^4x_1 \dots d^4x_n \, e^{i(p_1 x_1 + \dots + p_n x_n)} \cdot G^{(n)}(x_1, \dots, x_n)$$

- For each line there is a 4-momentum flowing through it

$$= \frac{i}{p^2 - m^2 + i\varepsilon}$$

- The momenta $p_1, \dots, p_n$ in $(*)$ flow <u>out</u> of the diagram

- Each vertex gives a factor

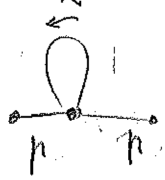


$$= -i \lambda_B \, (2\pi)^4 \, \delta(p_1 + p_2 + p_3 + p_4),$$

that is, 4-momentum is conserved at each vertex.

- Each momentum p flowing through internal lines is integrated with $\int \frac{d^4p}{(2\pi)^4}$

example: $\widetilde{G}^{(2)}(p, p') =$

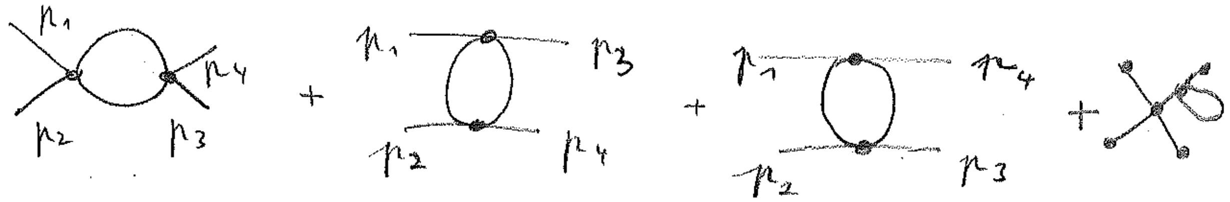


$+ \dots$

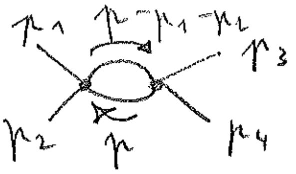
$$= -i \frac{\lambda}{2} (2\pi)^4 \delta(p + p') \left(\frac{i}{p^2 - m^2 + i\varepsilon}\right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}$$

Now compute something new:

$n = 4$, $q = 2$, connected



$+ \dots$



$$= (-i\lambda)^2 S \, (2\pi)^4 \, \delta(p_1 + \dots + p_4)$$

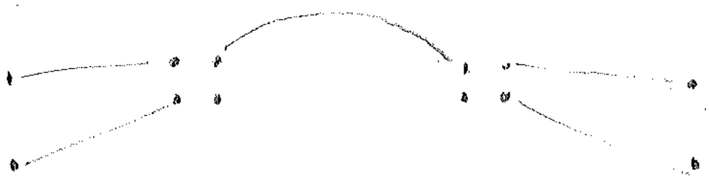
$$\frac{i}{p_1^2 - m^2 + i\varepsilon} \cdots \frac{i}{p_4^2 - m^2 + i\varepsilon}$$

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} \frac{i}{(p - p_1 - p_2)^2 - m^2 + i\varepsilon}$$

Determine the symmetry factor by hand:

prefactor: $\frac{1}{2}\left(\frac{1}{4!}\right)^2$ from $e^{iS_{int}} = 1 + iS_{int} + \frac{1}{2}(iS_{int})^2 + \mathcal{O}(\lambda_B^3)$

contractions which yield the same diagram:



$$2 \cdot (4 \cdot 3)^2 \cdot 2 = (4 \cdot 3 \cdot 2)^2$$

$$\Rightarrow S = 1/2$$

$$\tilde{G}^{(n)}(p_1, \dots, p_n) = (2\pi)^4 \, \delta(p_1 + \dots + p_n) \, G^{(n)}(p_1, \dots, p_{n-1})$$