3.6 Normalization factor

$$\mathcal{N}^{-1} = \int \mathcal{D}\varphi \, e^{iS} \quad , \qquad S = S_0 + S_{int}$$

expand in S_{int}:

$$\mathcal{N}^{-1} = \int \mathcal{D}\varphi \, e^{iS_0} \left(1 + i S_{int} + \mathcal{O}(\lambda_B^2)\right)$$

$$= \mathcal{N}_0^{-1}\left(1 + \mathcal{N}_0 \int \mathcal{D}\varphi \, e^{iS_0} \, i S_{int} + \mathcal{O}(\lambda_B^2)\right)$$

For any operator A define

$$\langle A \rangle_0 = \mathcal{N}_0^{-1} \int \mathcal{D}\varphi \, e^{iS_0} \, A$$

Then

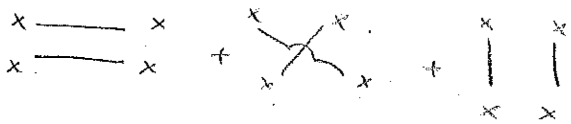
$$\mathcal{N} = \mathcal{N}_0 \left(1 + \langle i S_{int} \rangle_0 + \mathcal{O}(\lambda_B^2)\right)^{-1}$$

$$N = N_0 \left(1 - \langle i S_{int} \rangle_0 \right) + \mathcal{O}(\lambda_B^2)$$

The $\mathcal{O}(\lambda_B)$ correction to N is determined by the expectation value of S_{int} in the free QFT.

$$\langle i S_{int} \rangle_0 = -i \frac{\lambda_B}{4!} \int d^4x \, \langle \varphi^4(x) \rangle_0$$

$\langle \varphi^4 \rangle_0$ is a 4-point function that we know how to compute:

$\langle \varphi^4(x) \rangle_0 =$  $= 3 \left[\Delta_F(x-x) \right]^2$

$$= 3 \left[\Delta_F(0) \right]^2$$

all 3 contractions give the same result

All space-time points are the same, and the corresponding Feynman diagram is drawn like this:

 $= -i \frac{1}{8} \lambda_B \int d^4x \left[\Delta_F(0) \right]^2$

note that $\Delta_F(0) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}$ is

divergent at large k <u>UV-divergence</u>

The prefactor $\frac{1}{8} = \frac{1}{4!} \cdot 3$ is called <u>combinatorical factor</u>

$\uparrow$

from S_{int}

3.7 2-point function at $O(\lambda_B)$

$$G^{(2)}(x_1,x_2) = \langle T\varphi(x_1)\varphi(x_2)\rangle = N\int\mathcal{D}\varphi\,\varphi(x_1)\varphi(x_2)e^{iS}$$

we found

(*) $$N = N_0\,(1 - \langle iS_{int}\rangle_0)$$

$$N_0\int\mathcal{D}\varphi\,\varphi(x_1)\varphi(x_2)e^{iS} = N_0\int\mathcal{D}\varphi\,\varphi(x_1)\varphi(x_2)e^{iS_0}(1+iS_{int}) + O(\lambda_B^2)$$

$$= \Delta_F(x_1-x_2) + \underbrace{\langle\varphi(x_1)\varphi(x_2)\,iS_{int}\rangle_0}_{} + O(\lambda_B^2)$$

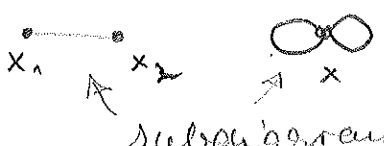
$$= -i\frac{\lambda_B}{4!}\int d^4x\,\langle\varphi(x_1)\varphi(x_2)\varphi^4(x)\rangle_0$$

free 6-point function with 4 fields at x

We have to sum all contractions of

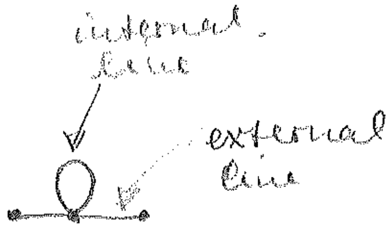
One obtains 2 types of Feynman diagrams:

(i) contract x_1 and x_2 → Feynman diagram



disconnected

(ii) contract x_1 and x

 → Feynman diagram 

connected

So we have

$$\langle\varphi(x_1)\varphi(x_2)\,iS_{int}\rangle_0 = \Delta_F(x_1-x_2)\langle iS_{int}\rangle_0$$
$$+ \langle\varphi(x_1)\varphi(x_2)\,iS_{int}\rangle_{0,c} \quad \text{'c' for connected}$$

(*) $\Rightarrow$

$$G^{(2)}(x_1,x_2) = (1 - i\langle S_{int}\rangle_0)\Big(\Delta_F(x_1 - x_2)$$

$$+ \Delta_F(x_1-x_2)\langle iS_{int}\rangle_0 + \langle\varphi(x_1)\varphi(x_2)\, iS_{int}\rangle_{0,c}\Big) + \mathcal{O}(\lambda_B^2)$$

$$= \Delta_F(x_1-x_2) + \langle\varphi(x_1)\varphi(x_2)\, iS_{int}\rangle_{0,c} + \mathcal{O}(\lambda_B^2)$$

The lines connected to x_i are called <u>external lines</u>. (Sub-) diagrams not connected to external lines are called vacuum diagrams. We found that diagrams with vacuum sub-diagrams do not contribute to $G^{(2)}$ at $\mathcal{O}(\lambda_B)$. One can show that this is a general result: diagrams with vacuum sub-diagrams do not contribute to n-point functions.

Now compute $\langle\varphi(x_1)\varphi(x_2)\, iS_{int}\rangle_{0,c} = -i\frac{\lambda}{4!}\int d^4x\, \langle\varphi(x_1)\varphi(x_2)\varphi^4(x)\rangle_{0,c}$

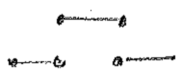
There are 4 possibilities to contract x_1 with a point at x. Each contraction gives the same result. $\to$ factor 4

4 $\cdot$ ()

3 possibilities for x_2 $\to$ factor 3

$4 \cdot 3$ ()

Now there's only 1 possibility left:

$4 \cdot 3$ ()

$$\langle\varphi(x_1)\varphi(x_2)\varphi^4(x)\rangle_{0,c} = 4\cdot 3\cdot\Delta_F(x_1-x)\,\Delta_F(0)\,\Delta_F(x-x_2)$$

$G^{(2)}(x_1, x_2) =$ $+$  $+ \mathcal{O}(\lambda_B^2)$

$$= \Delta_F(x_1 - x_2) - i\lambda \frac{1}{2} \int d^4x \, \Delta_F(x_1 - x) \Delta_F(0) \Delta_F(x - x_2)$$
$$+ \mathcal{O}(\lambda_B^2)$$

The point at x in  is called <u>vertex</u>

The Feynman diagrams contain all combinatorial factors from equivalent contractions, as well as the factor $\frac{-i\lambda}{4!}$

"coordinate space Feynman diagrams".

momentumspace:

$$\tilde{G}^{(2)}(p_1, p_2) = \int d^4x_1 \int d^4x_2 \, e^{i(p_1 x_1 + p_2 x_2)} G^{(2)}(x_1, x_2)$$

Translational invariance of the vacuum $\Rightarrow$

$$G^{(2)}(x_1, x_2) = G^{(2)}(x_1 - x_2, 0)$$

$$\tilde{G}^{(2)}(p_1, p_2) = \int d^4x_1 \int d^4x_2 \, e^{ip_1(x_1 + x_2)} e^{ip_2 x_2} \, G^{(2)}(x_1, 0)$$

$$= (2\pi)^4 \delta^4(p_1 + p_2) \, G^{(2)}(p_1)$$

with $G^{(2)}(p) = \int d^4x \, e^{ipx} \, G^{(2)}(x, 0)$

expansion in λ_B: $G^{(2)} = G_0^{(2)} + G_1^{(2)} + \mathcal{O}(\lambda_B^2)$

$$G_0^{(2)}(p_1) = \Delta_F(p_1)$$

$$G_1^{(2)}(p_1) = -i\frac{\lambda_B}{2}\int d^4x_1\, d^4x\, e^{ip_1x_1}\,\Delta_F(x_1-x)\,\Delta_F(0)\,\Delta_F(x)$$

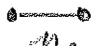
$$= -i\frac{\lambda_B}{2}\int d^4x\, d^4x_1\, e^{ip_1(x_1+x)}\,\Delta_F(x_1)\,\Delta_F(0)\,\Delta_F(x)$$

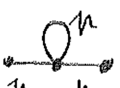
$$= -i\frac{\lambda_B}{2}\,\tilde{\Delta}_F(p_1)\,\Delta_F(0)\,\tilde{\Delta}_F(p_1)$$

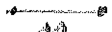
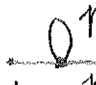
[To distinguish the momentum space Δ_F from the one in coordinate space, I have temporarily written it with a tilde]

$$\Delta_F(0) = \int\frac{d^4p}{(2\pi)^4}\,\tilde{\Delta}(p)$$

momentum-space Feynman diagram for $G^{(2)}(p_1)$:

 p_1 $= \dfrac{i}{p_1^2-m^2+i\varepsilon}$

 p, p_1 p_1 $= -i\lambda\,\dfrac{1}{2}\,\dfrac{i}{p_1^2-m^2+i\varepsilon}\displaystyle\int\frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\varepsilon}\,\frac{i}{p_1^2-m^2+i\varepsilon}$

$G^{(2)}(p_1) =$  p_1 $+$  p, p_1 p_1 $\quad p$ is called <u>loop momentum</u>

more about this later!