## 3.4 $Z[J]$ for free fields

[Ryder 6.1, 6.2]

consider real scalar field, $S = \int d^4x \left\{ -\frac{1}{2}\varphi(\partial^2 + m^2)\varphi \right\}$

$$Z[J] = N \int \mathcal{D}\varphi \exp\left\{ i \left[ S + \int d^4x \, J\varphi \right] \right\} \qquad \text{with}$$

$$N^{-1} = \int \mathcal{D}\varphi \, e^{iS}$$

The exponent contains at most 2 powers of $\varphi$. This is a Gauß integral, similar to the one for $\pi(x)$. There we solved it by shifting $\pi$.

Try something similar here: $\varphi \to \varphi + \varphi_0$
where $\varphi_0$ will be determined shortly.
Use $\int d^4x \, \varphi_0 (\partial^2 + m^2)\varphi = \int d^4x \, \varphi(\partial^2 + m^2)\varphi_0 \quad \Longrightarrow$

$$S + \int d^4x \, J\varphi = \int d^4x \left\{ -\frac{1}{2}\varphi(\partial^2 + m^2)\varphi - \varphi(\partial^2 + m^2)\varphi_0 - \frac{1}{2}\varphi_0(\partial^2 + m^2)\varphi_0 \right.$$
$$\left. + J\varphi + J\varphi_0 \right\}$$

Choose $\varphi_0$ such that

(*) $$(\partial^2 + m^2)\varphi_0 = J$$

Then the terms linear in $\varphi$ cancel, and the integral over $\varphi$ gives a factor $N^{-1}$ which then drops out in $Z$.

Solve (*) by FT:

$$\varphi_0(k) = \frac{1}{-k^2 + m^2} J(k)$$

For the inverse FT we need to know how to integrate around the pole at $k^2 = m^2$. Here we have to remember the $\varepsilon$'s. We had

$$\int d^4x \left\{ \frac{1}{1 - i\varepsilon} \dot{\varphi}^2 - (1 - i\varepsilon)\left( (\nabla\varphi)^2 + m^2\varphi^2 \right) \right\}$$

Thus we should replace

$$\frac{1}{k^2 - m^2} = \frac{1}{k_0^2 - \vec{k}^2 - m^2} \longrightarrow$$

$$\rightarrow \frac{1}{\frac{1}{1-i\varepsilon} k_0^2 - (1-i\varepsilon)(\vec{k}^2 + m^2)} = \frac{1}{k_0^2 - (1-i\varepsilon)^2(\vec{k}^2 + m^2)}$$

$$= \frac{1}{k^2 - m^2 + 2i\varepsilon E_{\vec{k}}^2}$$

we need not distinguish $2\varepsilon E_{\vec{k}}^2$ and $\varepsilon$.

We had $\Delta_F(k) = \frac{i}{k^2 - m^2 + i\varepsilon} \Rightarrow$

$$\varphi_0(k) = i\,\Delta_F(k)\,J(k)$$

$$\varphi_0(x) = i \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \Delta_F(k) \int d^4x'\, e^{ikx'} J(x')$$

(*) $$= i \int d^4x'\, \Delta_F(x - x')\, J(x')$$

$$Z[J] = \exp\left\{ i \int d^4x \left[ -\frac{1}{2} \varphi_0 \underbrace{(\partial^2 + m^2)\varphi_0}_{= J} + J\varphi_0 \right] \right\}$$

$$= \exp\left( i \int d^4x \, \frac{1}{2} J(x)\varphi_0(x) \right)$$

Insert (*) $\Rightarrow$

$$\boxed{Z[J] = \exp\left\{ -\frac{1}{2} \int d^4x \int d^4x'\, J(x)\, \Delta_F(x - x')\, J(x') \right\}}$$

From now on $\Delta_F$ always refers to the free 2-point function.

3.3

Now we can compute <u>n-point functions</u> from Z.

$$G^{(n)}(x_1,\dots,x_n) = \left(\frac{1}{i}\right)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J] \Big|_{J=0}$$

define $W[J]$ through $Z[J] = e^{W[J]}$

So we have

$$W[J] = -\frac{1}{2}\int d^4x \int d^4x' \, J(x)\, \Delta_F(x-x')\, J(x')$$

$$(\times) \quad Z = e^W = \sum_{l=0}^{\infty} \frac{W^l}{l!}$$ contains only even powers in $J$

$\Rightarrow$ $G^{(n)} = 0$ for odd $n$.

<u>2-point function</u> $G^{(2)}(x_1,x_2) = -\frac{\delta^2 W}{\delta J(x_1)\delta J(x_2)} = \Delta_F(x_1-x_2)$

<u>ok</u>.

<u>4-point function</u> only the $l=2$ term in $(\times)$ contributes

$$G^{(4)}(x_1,\dots,x_4) = \frac{\delta^4}{\delta J(x_1)\cdots} \frac{1}{2} W^2 = \frac{\delta^3}{\delta J(x_1)\cdots\delta J(x_3)} W \frac{\delta W}{\delta J(x_4)}$$

$$= \frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\left(\frac{\delta W}{\delta J(x_3)}\frac{\delta W}{\delta J(x_4)} + W \underbrace{\frac{\delta^2 W}{\delta J(x_3)\delta J(x_4)}}_{=-\Delta_F(x_3-x_4)}\right)$$

$$= \Delta_F(x_1-x_3)\,\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\,\Delta_F(x_2-x_3)$$

$$+ \Delta_F(x_1-x_2)\,\Delta_F(x_3-x_4)$$

For any even $n$:

$$\boxed{G^{(n)}(x_1,\dots,x_n) = \sum_{\text{pairings}} \Delta_F(x_{i_1}-x_{i_2}) \cdots \Delta_F(x_{i_{n-1}}-x_{i_n})}$$

<u>Wick's theorem</u>

## Feynman diagrams for free scalar fields

Wick's theorem has a simple graphical interpretation:

(i) draw a dot $\circ$ for each $x_i$

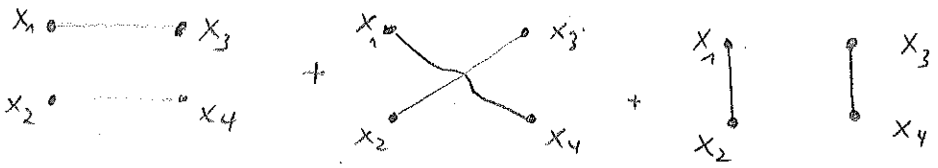
(ii) connect ('contract') all pairs of points with a line

(iii) a line from $x_i$ to $x_j$ gives a factor $\Delta_F(x_i - x_j)$

(iv) Sum these products over all possible pairings ('contractions')

$G^{(2)} =$  $= \Delta_F(x_1 - x_2)$ as before



$G^{(4)} =$ 



These are Feynman diagrams

The diagram for $G^{(2)}$ is called connected, those for $G^{(4)}$ disconnected.

$W[J]$ generates the connected $n$-point functions.

Without interactions, the only connected Green's function is $G^{(2)}$.

## 3.5 Interactions, perturbation theory

Consider again a real scalar, but now with

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$$

$$\mathcal{L}_0 = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2} m_B^2 \varphi^2$$

$m_B$: bare mass

$$\mathcal{L}_{int} = -\frac{1}{4!}\lambda_B \varphi^4$$

interaction Lagrangian

$\lambda_B$: bare coupling constant

$$Z[J] = N \int \mathcal{D}\varphi \exp\left\{ i\left(S_0 + S_{int} + \int d^4x\, J\varphi\right)\right\}$$ with

$$S_0 = \int d^4x\, \mathcal{L}_0, \qquad S_{int} = \int d^4x\, \mathcal{L}_{int}$$ and

$$N^{-1} = \int \mathcal{D}\varphi \exp\left\{ i\left(S_0 + S_{int}\right)\right\}$$

Expand $Z[J]$ in powers of $\lambda_B$. With the path integral this is straightforward because we only need to expand

$$e^{iS_{int}} = 1 + iS_{int} + \frac{1}{2}(iS_{int})^2 + \mathcal{O}(\lambda_B^3)$$

Start with expanding the

normalization factor

$$N^{-1} = \underbrace{\int \mathcal{D}\varphi\, e^{iS_0}}_{=:\, N_0^{-1}} + \int \mathcal{D}\varphi\, e^{iS_0}\, iS_{int} + \mathcal{O}(\lambda_B^2)$$

$$= N_0^{-1}\left[1 + N_0 \int \mathcal{D}\varphi\, e^{iS_0}\, iS_{int} + \mathcal{O}(\lambda_B^2)\right]$$

For any operator A define

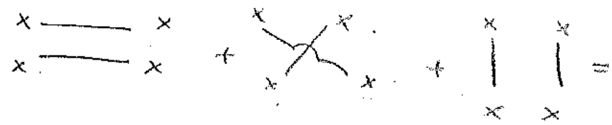
$$\langle A \rangle_0 := N_0 \int \mathcal{D}\varphi\, e^{iS_0} A$$

$$N = N_0 \left(1 - \langle i S_{int} \rangle_0 \right) + \mathcal{O}(\lambda_B^2)$$

The $\mathcal{O}(\lambda_B)$ correction to $N$ is determined by the expectation value of $S_{int}$ in the free QFT.

$$\langle i S_{int} \rangle_0 = -i \frac{\lambda_B}{4!} \int d^4x \, \langle \varphi^4(x) \rangle_0$$

$\langle \varphi^4 \rangle_0$ is a 4-point function that we know how to compute:

$\langle \varphi^4(x) \rangle_0 =$  $= 3\left[\Delta_F(x-x)\right]^2$

$$= 3 \left[\Delta_F(0)\right]^2$$

all 3 contractions give the same result

All space-time points are the same, and the corresponding Feynman diagram is drawn like this:

 $= -i \frac{1}{8} \lambda_B \int d^4x \left[\Delta_F(0)\right]^2$

note that $\Delta_F(0) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon}$ is divergent at large $k$ <u>UV-divergence</u>

The prefactor $\frac{1}{8} = \underbrace{\frac{1}{4!}}_{\text{from } S_{int}} \cdot 3$ is called <u>combinatorical factor</u>