3 Path integrals

3.1 Generating functional for n-point functions

Consider one Hermitean scalar field

$Z[J] := \langle 0| T\exp\left(i\int d^4x\, J(x)\varphi(x)\right)|0\rangle$

for any real c-number field J.

All n-point functions can be obtained from $Z[J]$ by functional differentiation.

$$G^{(n)}(x_1,\dots,x_n) = \frac{1}{(i)^n}\frac{\delta^n}{\delta J(x_1)\dots\delta J(x_n)} Z[J]\Big|_{J=0}$$

$$= \langle 0| T\varphi(x_1)\dots\varphi(x_n)|0\rangle$$

Z is called <u>generating functional</u>.

It has a simple physical interpretation:

Consider a system with Hamiltonian H, and add an additional term,

$H \to H' = H + V$.

with $V = -\int d^3x\, J(x)\varphi(x)$

which can be veiwed as an "external perturbation".

[recall that J is a classical field and φ is an operator] time evolution in the interaction picture respect to V; $|\psi(t)\rangle_I := e^{iHt} |\psi(t)\rangle_{\text{Schrödinger}}$:

$$i \frac{d}{dt} |\psi(t)\rangle_I = V(t) |\psi(t)\rangle_I$$

Solution:

$$|\psi(t)\rangle_I = T \exp\left\{ -i \int_{t_0}^{t} dt'\, V(t') \right\} |\psi(t_0)\rangle_I$$

$$= T \exp\left\{ +i \int_{t_0}^{t} dt' \int d^3x'\, J(x')\varphi(x') \right\} |\psi(t_0)\rangle_I$$

choose J such that $J(x) \to 0$ for $t \to \pm\infty$

initial condition: $|\psi(t_0)\rangle \to |0\rangle$ for $t_0 \to -\infty$

Then $Z[J]$ is the probability amplitude $\langle 0|0\rangle_J$ for the transition vacuum $\to$ vacuum in the presence of the perturbation J

How does one compute $Z[J]$?

In general, for interacting QFTs $Z[J]$ cannot be computed exactly.

One can use numerical methods or perturbation theory (PT). For perturbation theory there are several methods: (i) operator PT

(ii) path integral PT

3.2 Path integrals

reminder (see, e.g., Münster ch. 23, Sakurai, ch. 2.5):

1 particle moving in 1 dimension, coordinate q

probability amplitude to be at position q_f at time t_f if it was at q_i at time t_i:

(*) $\langle q_f | U(t_f, t_i) | q_i \rangle$

$$U(t_f, t_i) = T \exp\left(-i \int_{t_i}^{t_f} dt \, H(p, q, t)\right)$$

time evolution operator

This amplitude has the path integral representation

$$\langle q_f | U(t_f, t_i) | q_i \rangle =$$

(**)
$$= \int \mathcal{D}q \, \mathcal{D}p \, \exp\left\{ i \int_{t_i}^{t_f} dt \left[p(t) \dot{q}(t) - H(p(t), q(t), t) \right] \right\}$$

Here q and p are c-number functions, and one integrates over all paths $q(t)$ with $q(t_i) = q_i$, $q(t_f) = q_f$ and over all momentum space "paths" $p(t)$.

important special case: H is quadratic in p, like, e.g.

$$H(p,q) = \frac{p^2}{2} + W(q,t) \qquad (\text{particle with unit mass})$$

Then one can integrate out p in the path integral:

$$\int \mathcal{D}p \, \exp\left\{ i \int_{t_i}^{t_f} dt \left[p\dot{q} - \frac{1}{2} p^2 \right] \right\}$$

$$= \int \mathcal{D}p \, \exp\left\{ -\frac{i}{2} \int_{t_i}^{t_f} \left[(p-\dot{q})^2 - \dot{q}^2 \right] \right\}$$

substitute $p \to p + \dot{q}$

$$= \text{const.} \exp\left\{ \frac{i}{2} \int_{t_i}^{t_f} dt \, \dot{q}^2 \right\}$$

The constant is independent of q_i, q_f and can be absorbed in the normalization factor of the path integral. Then

$$\langle q_f | \mathcal{U}(t_f, t_i) | q_i \rangle = \int \mathcal{D}q \, e^{iS}$$

with $$S[q] = \int_{t_i}^{t_f} dt \left(\frac{1}{2}\dot{q}^2 - W \right) = \int_{t_i}^{t_f} dt \, L$$

and the same boundary conditions as before.

In <u>scalar field theory</u>, the amplitude $(*)$ is replaced by

$$\langle \varphi_f | \mathcal{U}(t_f, t_i) | \varphi_i \rangle$$

which is the amplitude that a field configuration $\varphi_i(\vec{x})$ at time t_i turns into the field configuration $\varphi_f(\vec{x})$ at time t_f.

Following the same steps that lead to (**) gives the path integral representation

$$\boxed{\begin{aligned}&\langle \varphi_f | \mathcal{U}(t_f, t_i) | \varphi_i \rangle \\ &= \int_{\substack{\varphi(t_i)=\varphi_i \\ \varphi(t_f)=\varphi_f}} \mathcal{D}\varphi\, \mathcal{D}\pi \, \exp\left\{ i \int_{t_i}^{t_f} dt \int d^3x \left[\pi(x)\, \dot{\varphi}(x) - \mathcal{H}(\pi(x), \varphi(x), x) \right] \right\}\end{aligned}}$$

Now one integrates over all field configurations with $\varphi(t_i, \vec{x}) = \varphi_i(\vec{x})$, $\varphi(t_f, \vec{x}) = \varphi_f(\vec{x})$, and over all conjugate momenta $\pi(t, \vec{x})$.

We are interested in the special case

(*) $\mathcal{H}(\pi, \varphi) = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + V(\varphi)$

Then we can integrate out π:

$$\int \mathcal{D}\pi \exp\left\{ i \int_{t_i}^{t_f} dt \int d^3x\, \underbrace{\left[\pi\dot{\varphi} - \tfrac{1}{2}\pi^2 \right]}_{= -\frac{1}{2}(\pi - \dot{\varphi})^2 + \frac{1}{2}\dot{\varphi}^2} \right\}$$

$$= \text{const} \exp\left\{ i \int_{t_i}^{t_f} dt \int d^3x \, \tfrac{1}{2}\dot{\varphi}^2 \right\} \quad \Rightarrow$$

$$\boxed{\langle \varphi_f | \mathcal{U}(t_f, t_i) | \varphi_i \rangle = \int_{\substack{\varphi(t_i)=\varphi_i \\ \varphi(t_f)=\varphi_f}} \mathcal{D}\varphi \, e^{iS}}$$

with $S = \int_{t_i}^{t_f} dt \int d^3x \, \mathcal{L}$, $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V(\varphi)$

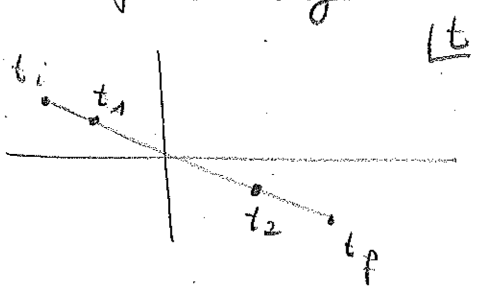
remark: when we later treat fermions, we will not integrate out the canonical momenta.

3.3 Path integral for $Z[J]$

We had $Z[J]$ = vacuum → vacuum amplitude for the Hamiltonian

$$H' = H - \int d^3x \, J \varphi \quad \text{, with } H \text{ time-independent.}$$

So far, we had a path integral for amplitudes for $\varphi_i \to \varphi_f$. Now we'll see how one can pick out the vacuum: Let t be on the following contour in the complex time plane:



$t_{i,f} = \mp (1 - i\varepsilon) T$

$\varepsilon > 0$, small

Let $J(t, \underline{x}) = 0$ for $\operatorname{Re} t < \operatorname{Re} t_1$, $\operatorname{Re} t > \operatorname{Re} t_2$

$$\langle \varphi_f | \mathcal{U}(t_f, t_i) | \varphi_i \rangle = \langle \varphi_f | \underbrace{\mathcal{U}(t_f, t_2)}_{= e^{-iH(t_f - t_2)}} \mathcal{U}(t_2, t_1) \underbrace{\mathcal{U}(t_1, t_i)}_{= e^{-iH(t_1 - t_i)}} | \varphi_i \rangle$$

Let $\{ |n\rangle \}$ be a complete set of H-eigenstates,

$$H|n\rangle = E_n |n\rangle , \qquad 1 = \sum_n |n\rangle\langle n|$$

Choose H such that $H|0\rangle = 0$

$$\langle \varphi_f | \mathcal{U}(t_f, t_i) | \varphi_i \rangle = \sum_{n_1 n_2} e^{-iE_{n_2}(t_f - t_2)} \langle \varphi_f | n_2 \rangle$$

$$\cdot \langle n_2 | \mathcal{U}(t_2, t_1) | n_1 \rangle \langle n_1 | \varphi_i \rangle e^{-iE_{n_1}(t_1 - t_i)}$$

make T very large.

$$e^{-iE_n t_f} = e^{E_n t_i} = e^{-iE_n T} e^{-\varepsilon E_n T} \to 0$$

for $T \to \infty$, except for $n = 0$. $\Rightarrow$

For $T \to \infty$ only $|0\rangle$ contributes to the amplitude:

$$\lim_{T\to\infty} \langle \varphi_f | \mathcal{U}(t_f, t_i) | \varphi_i \rangle = \langle \varphi_f | 0 \rangle \langle 0 | \varphi_i \rangle \langle 0 | \mathcal{U}(t_f, t_i) | 0 \rangle$$

$$= \langle \varphi_f | 0 \rangle \langle 0 | \varphi_i \rangle \, Z[J]$$

We already know how to write this as a path integral:

$$\lim_{T\to\infty} \langle \dots \rangle = \int \mathcal{D}\varphi \, \mathcal{D}\pi \, \exp\left\{ i \int dt \int d^3x \, [\pi \dot\varphi - \mathcal{H} + J\varphi] \right\}$$

$$Z[0] = 1 \Rightarrow \langle \varphi_f | 0 \rangle \langle 0 | \varphi_i \rangle = \int \mathcal{D}\varphi \, \mathcal{D}\pi \, \exp\left\{ i \int d^4x \, [\pi\dot\varphi - \mathcal{H}] \right\} \Rightarrow$$

$$\boxed{Z[J] = \frac{\int \mathcal{D}\varphi \, \mathcal{D}\pi \, \exp\left(i \int d^4x \, [\pi\dot\varphi - \mathcal{H} + J\varphi] \right)}{\int \mathcal{D}\varphi \, \mathcal{D}\pi \, \exp\left(i \int d^4x \, [\pi\dot\varphi - \mathcal{H}] \right)}}$$

If $\mathcal{H}$ is quadratic in π, we can again integrate out π.
$\Rightarrow$

$$\boxed{Z[J] = \frac{\int \mathcal{D}\varphi \, \exp\left(i \int d^4x \, [\mathcal{L} + J\varphi] \right)}{\int \mathcal{D}\varphi \, \exp\left(i \int d^4x \, \mathcal{L} \right)}}$$

Here the time integral is along the tilted path in the complex plane. We can re-write our result in terms of an integral over real time if we replace

$$t \to (1 - i\varepsilon) t, \quad \partial_t \to \frac{1}{1 - i\varepsilon} \partial_t$$

Then for $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V(\varphi)$:

$$\int d^4x\,(\mathcal{L} + J\varphi) \to \int d^4x\,(1-i\varepsilon)\left\{\frac{1}{(1-i\varepsilon)^2}\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\nabla\varphi)^2 - V + J\varphi\right\}$$

$$= \int d^4x\left\{\frac{1}{1-i\varepsilon}\frac{1}{2}\dot{\varphi}^2 + (1-i\varepsilon)\left[-\frac{1}{2}(\nabla\varphi)^2 - V(\varphi) + J\varphi\right]\right\}$$

Usually we do not write the $i\varepsilon$'s explicitly. However, sometimes we will need them, and we have to re-introduce them at the appropriate places.