2.7 Lehmann-Symanzik-Zimmermann (LSZ) formula

[Coleman pg 139, Srednicki 5]

Now we are ready to build in and out states. Let w_1, w_2 be normalized wave packets moving in different directions.

For any normalizable $|\psi\rangle$:

$$\lim_{t\to\infty} \langle\psi|a_1^\dagger(t)|w_2\rangle = \langle\psi|w_1 w_2 \text{ out}\rangle$$

$$\lim_{t\to-\infty} \langle\psi|a_1^\dagger(t)|w_2\rangle = \langle\psi|w_1 w_2 \text{ in}\rangle$$

check: For $t\to\pm\infty$ the wave packets are widely separated. For an observer near $\mathrm{supp}(W_1)$ applying $a_1^\dagger$ to $|w_2\rangle$ is like applying it to the vacuum.

We have already achieved our main goal, which was to write the scattering amplitude in terms of field operators:

$$\langle w_3 w_4 \text{ out}|w_1 w_2 \text{ in}\rangle$$

$$= \lim_{t_1,t_2\to-\infty} \lim_{t_3,t_4\to\infty} \langle 0|a_4(t_4)\, a_3(t_3)\, a_1^\dagger(t_1)\, a_2^\dagger(t_2)|0\rangle$$

Define the linear operator S through

$$S^\dagger |a \text{ in}\rangle = |a \text{ out}\rangle$$

$\forall$ in states $|a \text{ in}\rangle$, called <u>S-matrix</u> or <u>S-operator</u>.

$$\langle w_3 w_4 \text{ out}|w_1 w_2 \text{ in}\rangle = \langle w_3 w_4 \text{ in}|S|w_1 w_2 \text{ in}\rangle$$

We will see that

$$(*)\quad \boxed{\begin{aligned}&\langle w_3 w_4 \text{ in} | S - 1 | w_1 w_2 \text{ in}\rangle \\ &= \int d^4x_1 \ldots d^4x_4\, W_3^*(x_3)\, W_4^*(x_4)\, W_1(x_1)\, W_2(x_2) \\ &\quad (i Z^{-1/2})^4 \prod_{a=1}^{4} (\partial_a^2 + m^2) \langle 0 | T \varphi(x_1) \cdots \varphi(x_4) | 0\rangle\end{aligned}}$$

with $\partial_a^2 = \partial^2/(\partial x^\mu_a \partial x_{a\mu})$. Check:

$$i\int d^4x\, W(\partial^2 + m^2)\varphi = i\int d^4x \{ W\partial_t^2 \varphi + \varphi(-\Delta + m^2) W\}$$

$(\partial^2 + m^2) W = 0$ because it's a superposition of $e^{-ik\cdot x}$ with $k_0 = E_{\vec{k}}$ $\Longrightarrow$

$$(-\Delta + m^2) W = -\partial_t^2 W \qquad \Longrightarrow$$

$$i\int d^4x\, W(\partial^2 + m^2)\varphi = i\int d^4x\, \partial_t (W\partial_t\varphi - \varphi\partial_t W)$$

$$= i\int dt\, \partial_t \int d^3x\, W \overleftrightarrow{\partial_t} \varphi = -\sqrt{Z}\int dt\, \partial_t\, a^\dagger(t)$$

$$= \sqrt{Z}\left(\lim_{t\to -\infty} - \lim_{t\to\infty}\right) a^\dagger(t)$$

Hermitean conjugate $\Longrightarrow$

$$i\int d^4x\, W^*(\partial^2 + m^2)\varphi = \sqrt{Z}\left(\lim_{t\to\infty} - \lim_{t\to-\infty}\right) a(t)$$

$$\text{RHS}(*) = \left(\lim_{t_4\to\infty} - \lim_{t_4\to-\infty}\right)\left(\lim_{t_3\to\infty} - \lim_{t_3\to-\infty}\right)$$

$$\left(\lim_{t_1\to-\infty} - \lim_{t_1\to\infty}\right)\left(\lim_{t_2\to-\infty} - \lim_{t_2\to\infty}\right)$$

$$\langle 0 | T\, a_3(t_3)\, a_4(t_4)\, a_1^\dagger(t_1)\, a_2^\dagger(t_2) | 0\rangle$$

When $t_2 \to \infty$, t_2 is the largest time, and the time ordering puts a_2^+ on the left. This contribution vanishes since $\lim_{|t|\to\infty} \langle 0 | a^+(t) | \psi \rangle = 0$

$$RHS(*) = \left(\lim_{t_4\to\infty} - \lim_{t_4\to-\infty}\right)\left(\lim_{t_3\to\infty} - \lim_{t_3\to-\infty}\right)$$

$$\left(\lim_{t_1\to-\infty} - \lim_{t_1\to\infty}\right) \langle 0 | T a_3(t_3) a_4(t_4) a_1^+(t_1) | w_2 \rangle$$

For the same reason only $\lim_{t_1\to-\infty} a_1^+(t_1)$ contributes $\Rightarrow$

$$RHS(*) = \left(\lim_{t_4\to\infty} - \lim_{t_4\to-\infty}\right)\left(\lim_{t_3\to\infty} - \lim_{t_3\to-\infty}\right)$$

$$\langle 0 | T a_3(t_3) a_4(t_4) | w_1 w_2 \text{ in} \rangle$$

$$= \left(\lim_{t_4\to\infty} - \lim_{t_4\to-\infty}\right)\Big\{\langle w_3 | a_4(t_4) | w_1 w_2 \text{ in} \rangle - \langle 0 | a_4(t_4) | \psi \rangle \Big\}$$

with
$$|\psi\rangle = \lim_{t_3\to-\infty} a_3(t_3) | w_1 w_2 \text{ in} \rangle$$

Now use

$$\lim_{t\to\pm\infty} \langle 0 | a(t) | \psi \rangle = \langle w | \psi \rangle \quad \text{for both } t \to \pm\infty$$

so that $\left(\lim_{t_4\to\infty} - \lim_{t_4\to-\infty}\right) \langle 0 | a_4(t_4) | \psi \rangle = 0$.

$$RHS(*) = \langle w_3 w_4 \text{ out} | w_1 w_2 \text{ in} \rangle - \langle w_3 w_4 \text{ in} | w_1 w_2 \text{ in} \rangle$$

$$= \langle w_3 w_4 \text{ in} | S - 1 | w_1 w_2 \text{ in} \rangle \qquad \square$$

Define

$$G_R(x_1,\dots,x_4) := (Z^{-\frac{1}{2}})^4 \langle 0 | T\varphi(x_1)\dots\varphi(x_4)|0\rangle$$

<u>renormalized 4-point function</u>

In $(*)$, use the definition of W_a $\Rightarrow$

$$\langle w_3 w_4 \text{ in} | S-1 | w_1 w_2 \text{ in}\rangle = \int d^4x_1 \dots \int d^4 x_4$$

$$\int d^3k_1\, w_1(\vec{k}_1) e^{-ik_1x_1} \dots \int d^3k_4\, w_4^*(\vec{k}_4) e^{ik_4x_4}\Big|_{k_a^0 = E_{\vec{k}_a}}$$

$$i^4 \prod_{a=1}^{4} (\partial_a^2 + m^2)\, G_R(x_1,\dots,x_4)$$

$$= \int d^3k_1\, w_1(\vec{k}_1)\dots\int d^3k_4\, w_4^*(\vec{k}_4)$$

$$i^4 \lim_{k_a^0\to E_{\vec{k}_a}} \left[\prod_{a=1}^{4}(-k_a^2+m^2)\right]\tilde{G}_R(-k_1,-k_2,k_3,k_4)$$

This expression shows that $\tilde{G}_R(-k_1,\dots,k_4)$ has poles at $k_a^2 = m^2$, and that their residues determine the S-matrix elements.

The RHS is a convolution of the momentum-space wave functions with the <u>momentum space S-matrix</u>

$$\langle w_3 w_4 \text{ in}|S-1|w_1 w_2 \text{ in}\rangle = \int d^3k_1\, w_1(\vec{k}_1)\dots\int d^3k_4\, w_4^*(\vec{k}_4) \cdot \langle \vec{k}_3\vec{k}_4 | S-1 | \vec{k}_1\vec{k}_2\rangle$$

for which we found

$$\boxed{\langle \vec{k}_3\vec{k}_4|S-1|\vec{k}_1\vec{k}_2\rangle = i^4 \lim_{k_a^0\to E_{\vec{k}_a}} \left[\prod_{a=1}^{4}(-k_a^2+m^2)\right]\tilde{G}_R(-k_1,-k_2,k_3,k_4)}$$

This is the famous <u>LSZ formula</u>

2.8 Cross sections

[Sterman 4.4]

Usually one is interested in probabilities to find a definite number of particles with definite momenta.

projection onto n-particle state

$$P(n\text{ particles}) = \langle w_1 w_2 \text{ in} | \underbrace{\mathbb{1}_{n\text{ particles}}} | w_1 w_2 \text{ in}\rangle$$

$$= \prod_{a=1}^{n}\left[\int_{\vec{l}_a}\frac{1}{(2\pi)^3 2 l_a^0}\right] |l \text{ out}\rangle\langle l \text{ out}| \qquad \text{with } l = (l_1, \dots, l_n)$$

$k := (k_1, k_2)$, $\int_{\vec{k}} := \int d^3k$, all 4 momenta on shell.

$$\left[\prod_{a=1}^{n}\frac{(2\pi)^3 2 l_a^0}{d^3 l_a}\right] dP = |\langle w_1 w_2 \text{ in} | l \text{ out}\rangle|^2$$

$$= \left[\prod_{b=1}^{2}\int_{\vec{k}_b \vec{k}_b'} w_b(\vec{k}_b)\, w_b^*(\vec{k}_b')\right] \langle k' \text{ in} | l \text{ out}\rangle\langle l \text{ out} | k \text{ in}\rangle$$

write $S - 1 = i\,T$

$$\langle f | T | i\rangle = (2\pi)^4\, \delta(p_i - p_f)\, \mathcal{M}$$

$\left[\text{for } \{\vec{l}_1, \vec{l}_2\} \neq \{\vec{k}_1, \vec{k}_2\}\right]$

$$\langle l \text{ out} | k \text{ in}\rangle = i(2\pi)^4\, \delta(k_1 + k_2 - \sum_j l_j)\, \mathcal{M}(k, l)$$

$$|\langle w_1 w_2 \text{ in} | l \text{ out}\rangle|^2 = \left[\prod_{b=1}^{2}\int_{\vec{k}_b \vec{k}_b'} w_b(\vec{k}_b)\, w_b^*(\vec{k}_b')\right]$$

$$\cdot\, \mathcal{M}(k, l)\, \mathcal{M}^*(k', l)$$

$$\cdot\, (2\pi)^4\, \delta(k_1 + k_2 - k_1' - k_2') \quad (2\pi)^4\, \delta(k_1 + k_2 - \sum_a l_a)$$

$$(2\pi)^4 \delta(k_1 + k_2 - k_1' - k_2') = \int d^4x \, \exp\left(-i\,[k_1 + k_2 - k_1' - k_2'] \cdot x\right)$$

If it were not for the $\vec{k}$-dependence of T, the $\vec{k}_b, \vec{k}'_b$ integrals would give

$$W_b(x) = \int_{\vec{k}} e^{-ik\cdot x} \, w_b(\vec{k})$$

We are interested in wave functions $w_b(\vec{k})$ which are sharply peaked around $\vec{k}_b = \vec{p}_b$, such that we can replace $\vec{k}_b \to \vec{p}_b$ in $\mathcal{M}$.

We also assume that we can make this replacement in the remaining delta function.

$$|\langle w.\text{in} | \ell\ \text{out}\rangle|^2 = \int d^4x \left[\prod_{b=1}^{2} |W_b(x)|^2\right] |\mathcal{M}(p,\ell)|^2 (2\pi)^4 \delta(p_1 + p_2 - \sum_a \ell_a)$$

normalization of W_b:

$$1 = \langle w_b | w_b \rangle = \int_{\vec{k}\,\vec{k}'} w_b(\vec{k})\, w_b^*(\vec{k}')\, (2\pi)^3 2k^0 \,\delta(\vec{k} - \vec{k}')$$

$$= \int_{\vec{k}} (2\pi)^3 2k^0 |w_b(\vec{k})|^2 \simeq (2\pi)^3 2p_b^0 \int_{\vec{k}} |w_b(\vec{k})|^2$$

$$\int d^3x \, |W_b(x)|^2 = \int_{\vec{k}\,\vec{k}'} \underbrace{\int d^3x \, e^{-ik\cdot x} \, e^{ik'\cdot x}}_{=(2\pi)^3 \delta(\vec{k} - \vec{k}')} w_b(\vec{k})\, w_b^*(\vec{k}')$$

$$= (2\pi)^3 \int_{\vec{k}} |w_b(\vec{k})|^2 \simeq \frac{1}{2p_b^0}$$

$\rho :=$ 1-particle density $\Rightarrow \int d^3x \, \rho(x) = 1$

If $|W_i(x)| \simeq$ const within $\mathrm{supp}(W_i) \Rightarrow$

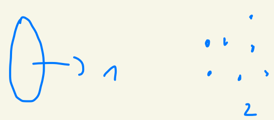
$$|W_i(x)|^2 \simeq \frac{\rho(x)}{2 p_i^0}$$

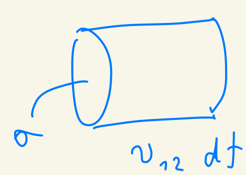
To identify a quantity which is determined by the interaction only,

Consider 2 classical beams with densities ρ_1, ρ_2

$\vec{v}_1 \rightarrow \quad \leftarrow \vec{v}_2$ relative velocity $v_{12} = |\vec{v}_1 - \vec{v}_2|$

Let particle 1 have a cross sectional area σ and consider the frame where particles 2 are at rest.





$$\# \text{ collisions} = \left(\rho_2 \, \sigma \, v_{12} \, dt\right)\left(d^3x \, \rho_1\right)$$

This motivates the def. of

$$\sigma := \frac{\# \text{ collisions}}{v_{12} \, d^3x \, dt \, \rho_1 \, \rho_2}$$

$$\boxed{d\sigma = \prod_{i=1}^{2}\left(\frac{1}{2p_i^0}\right)\frac{1}{v_{12}}\prod_{j=1}^{n}\left(\frac{d^3 l_j}{(2\pi)^3 2 l_j^0}\right)\left|\mathcal{M}(p_1, p_2, l_1, \dots, l_n)\right|^2 (2\pi)^4 \delta\Big(p_1 + p_2 - \sum_j l_j\Big)}$$