2.5 In and out states

In scattering experiments all particles are far away from each other for $t \to \pm\infty$. If the interaction has finite range, the particles then behave like in 1-particle states.

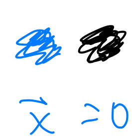
Here it is important that the particles are somewhat localized. Thus they cannot be in momentum eigenstates. Instead they must be in wave packets

for 1 particle: $\quad |w\rangle = \int d^3k \; |\vec{k}\rangle \, w(\vec{k})$

w: momentum space wave function

We now want to describe the scattering of 2 particles. Let $|w_1\rangle$ and $|w_2\rangle$ be 1-particle states which in the far past were moving towards each other

and for $t = 0$ are both localized near $\vec{x} = 0$



To describe the scattering of 2 particles we consider a state $|w_1 w_2 \text{ in}\rangle$ which for $t \to -\infty$ looks like $|w_1\rangle$ in the region where one particle is localized and like $|w_2\rangle$ in the region where the other particle is localized and like $|0\rangle$ otherwise.

$t \to -\infty$:

$|w_1 w_2 \text{ in}\rangle$:  <u>in state</u>

If the particles interact, then for $t \to +\infty$ the state $|w_1 w_2 \text{ in}\rangle$ will in general <u>not</u> look like $|w_1\rangle$ in some region and like $|w_2\rangle$ in some other region!

One can have particle production, so that for $t \to \infty$ the state $|w_1 w_2 \text{ in}\rangle$ may not even be a 2-particle state, but instead a superposition of 2-particle, 3-particle, ... states.

Suppose we want to compute the probability amplitude to find exactly 2 particles in wave packets w_3, w_4 for $t \to \infty$. For this define <u>the out state</u> as follows:

$t \to \infty$

$|w_3 w_4 \text{ out}\rangle$: 

Then the amplitude we are looking for is

$$\langle w_3 w_4 \text{ out} | w_1 w_2 \text{ in} \rangle$$

remark: without interactions

$$|w_3 w_4 \text{ out}\rangle = |w_3 w_4 \text{ in}\rangle$$

2.6 Normalized 1-particle states

[Coleman pg 159, Srednicki 5]

Let us now try to relate the in and out states to field operators. The first step is to do this for 1-particle wave packets

$$|w\rangle = \int d^3k \, |\vec{k}\rangle w(\vec{k})$$

This is easy for free fields: $|\vec{k}\rangle = a^\dagger_{\vec{k}} |0\rangle$,

$$|w\rangle = a^\dagger |0\rangle \quad \text{with} \quad a^\dagger := \int d^3k \, a^\dagger_{\vec{k}} \, w(\vec{k})$$

In chapter 1 we found that with $k^0 = E_{\vec{k}}$

$$a^\dagger_{\vec{k}} = \int d^3x \, e^{-ikx} (E_{\vec{k}} \varphi - i\dot{\varphi}) = i \int d^3x \, \varphi \overleftrightarrow{\partial_t} e^{-ikx}$$

which was independent of t. $\Rightarrow$

$$a^\dagger = i \int d^3x \, \varphi(x) \overleftrightarrow{\partial_t} W(x)$$

with

$$W(x) := \int d^3k \, w(\vec{k}) \, e^{-ikx} \Big|_{k^0 = E_{\vec{k}}}$$

Here a^+ is time independent as well.

Now try to do something similar for interacting fields. Use the renormalized field

$$\varphi' := \frac{1}{\sqrt{\hat{Z}}} \varphi$$

and define

$$a^+(t) := i \int d^3x \, \varphi'(x) \overleftrightarrow{\partial_t} W(x)$$

Haag-Ruelle operator

with the same W as in (*)

In general, $a^+(t)$ will now depend on t. But I claim that

(*) $$\boxed{\lim_{t \to \pm\infty} a^+(t) |0\rangle = |w\rangle}$$

check: without loss of generality we may assume that $\langle 0 | \varphi'(x) | 0 \rangle = 0$. [If not: $\langle 0 | \varphi'(x) | 0 \rangle = \langle 0 | e^{iPx} \varphi'(0) e^{-iPx} | 0 \rangle = \langle 0 | \varphi'(0) | 0 \rangle$. Now re-define $\varphi'(x) \to \varphi'(x) - \langle 0 | \varphi'(0) | 0 \rangle$] $\Rightarrow$

$\langle 0 | a^+(t) | 0 \rangle = 0$, that is, $a^+|0\rangle$ and $a|0\rangle$ are both orthogonal to $|0\rangle$.

next show that $a^\dagger|0\rangle$ has non-zero overlap with the 1-particle state $|\vec{p}\rangle$

$$\langle\vec{p}|a^\dagger(t)|0\rangle = i\int d^3x \langle\vec{p}|\varphi'\dot{W} - W\dot{\varphi}'|0\rangle$$

$$= i\int d^3x \int d^3k\, w(\vec{k})\, e^{-ikx} \underbrace{\langle\vec{p}|\varphi'(-ik^0) - \dot{\varphi}'|0\rangle}\Big|_{k^0=E_{\vec{k}}}$$

$$= (-ik^0 - \partial_t)\underbrace{\langle\vec{p}|\varphi'(x)|0\rangle}_{=e^{ipx}\ (p^0=E_{\vec{p}})} = (-i)(E_{\vec{k}}+E_{\vec{p}})e^{ipx}$$

$$\int d^3x\, e^{i(\vec{k}-\vec{p})\vec{x}} = (2\pi)^3\,\delta(\vec{k}-\vec{p}) \qquad \Longrightarrow$$

$$\langle\vec{p}|a^\dagger(t)|0\rangle = (2\pi)^3\, 2E_{\vec{p}}\, w(\vec{p}) = \langle\vec{p}|w\rangle$$

Furthermore,

$$\langle 0|a^\dagger|\vec{p}\rangle = \langle\vec{p}|a|0\rangle^* = 0$$

since e^{ipx} gets replaced by e^{-ipx} and $E_{\vec{k}}$ by $-E_{\vec{k}}$ in the calculation above.

Therefore $a^\dagger(t)$ acts like a creation operator. However, in general it can also generate more than a single particle. That's why need the limit in $(*)$.

Let $|p\lambda\rangle$ be a multi-particle state (more than one) with 4-momentum p^μ (recall that now $p^0 > E_{\vec{p}}$). Now we have

$$\langle p\lambda|\varphi'(x)|0\rangle = e^{ipx}\langle p\lambda|\varphi'(0)|0\rangle \qquad \Longrightarrow$$

$$\langle p\lambda|a^\dagger(t)|0\rangle = (2\pi)^3\, w(\vec{p})\,(p^0+E_{\vec{p}})\,e^{-i(E_{\vec{p}}-p^0)t}\langle p\lambda|\varphi'(0)|0\rangle$$

This does not vanish and the limit $t\to\pm\infty$ does not exist.

However, all we need to know are matrix elements with normalizable multi-particle states $|\chi\rangle$:

$$\langle\chi|a^{\dagger}(t)|0\rangle = \sum\!\!\!\!\!\!\int_{\lambda\vec{p}} \langle\chi|p\lambda\rangle\langle p\lambda|a^{\dagger}(t)|0\rangle$$

$$= \sum\!\!\!\!\!\!\int_{\lambda\vec{p}} \langle\chi|p\lambda\rangle (2\pi)^3\, w(\vec{p})(p^0+E_{\vec{p}})\, e^{-i(E_{\vec{p}}-p^0)t}\langle p\lambda|\varphi'(0)|0\rangle$$

For larger and larger $|t|$ the exponential oscillates more and more rapidly when integrating over $\vec{p}$. $\Longrightarrow$

$$\lim_{|t|\to\infty} \langle\chi|a^{\dagger}(t)|0\rangle = 0$$

similarly (this we will need later)

$$\lim_{|t|\to\infty} \langle 0|a^{\dagger}(t)|\chi\rangle = 0$$

So for any normalizable state $|\psi\rangle$:

$$\lim_{t\to\pm\infty} \langle\psi|a^{\dagger}(t)|0\rangle = \langle\psi|w\rangle \qquad \square$$

We will also need that

$$\lim_{|t|\to\infty} \langle 0|a^{\dagger}(t)|\psi\rangle = 0$$

which can be shown similarly.