2.3 Energy spectrum

assume Poincaré invariance and $P^\mu |0\rangle = 0$

$[P^0, \vec{P}] = 0 \quad \Rightarrow \quad H$ and $\vec{P}$ are simultaneously diagonalizable

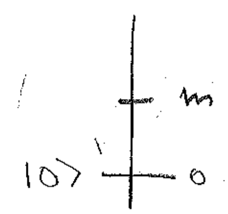
In general the eigenvalues are highly degenerate, label them by the parameter λ.

$$H\,|\vec{k}\,\lambda\rangle = E_{\vec{k}\lambda}\,|\vec{k}\,\lambda\rangle \quad , \quad \vec{P}\,|\vec{k}\,\lambda\rangle = \vec{k}\,|\vec{k}\,\lambda\rangle$$

normalization:

$$\langle \vec{k}\,\lambda\,|\,\vec{k}'\lambda'\rangle = 2E_{\vec{k}\lambda}\,(2\pi)^3\,\delta(\vec{k}-\vec{k}')\,\delta_{\lambda\lambda'}$$

Assume that there is a <u>mass gap</u>



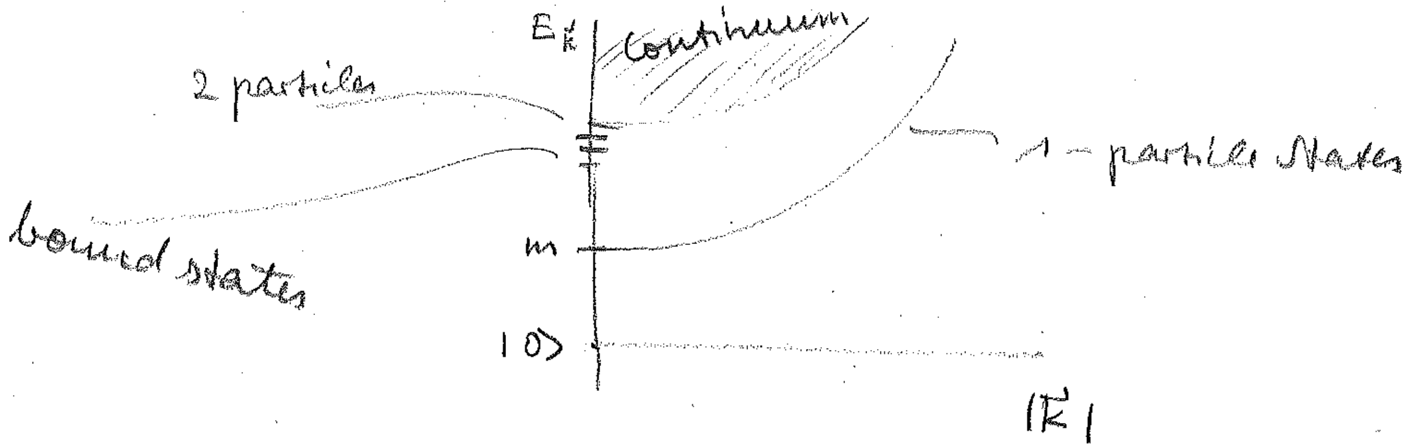
For every $|\vec{k}\,\lambda\rangle$ there is a LT Λ such that

$$(*) \qquad |\vec{k}\,\lambda\rangle = U(\Lambda)\,|\vec{0}\,\lambda\rangle$$

Define M_λ through $\quad H\,|\vec{0}\,\lambda\rangle = M_\lambda\,|\vec{0}\,\lambda\rangle$

Then $\quad E_{\vec{k}\lambda} = \sqrt{\vec{k}^2 + M_\lambda^2}\,.$

for a 1-particle state: $M_\lambda = m$



2.4 Propagator [Peskin 7.1]

Consider a real scalar field.

$$\Delta_F(x) = \langle 0 | T\varphi(x)\varphi(0)|0\rangle$$

for free fields: $\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ikx}$

$\Delta_F(k)$ has poles at $k^0 = \pm E_{\vec{k}}$

We will see that is also true with interactions.

$P^\mu|0\rangle = 0$, $|0\rangle$ invariant under translations

$$\Delta_F(x) = \Theta(t)\langle 0|\varphi(x)\varphi(0)|0\rangle + \Theta(-t)\underbrace{\langle 0|\varphi(0)\varphi(x)|0\rangle}_{=\langle 0|\varphi(-x)\varphi(0)|0\rangle}$$

$$= \Theta(t)\langle 0|\varphi(x)\varphi(0)|0\rangle + (x \to -x)$$

$\varphi(x) = e^{iPx}\varphi(0)e^{-iPx} \implies$

$$\langle 0|\varphi(x)\varphi(0)|0\rangle = \langle 0|\varphi(0)e^{-iPx}\varphi(0)|0\rangle$$

insert a complete set of states

$$1 = |0\rangle\langle 0| + \sum_\lambda \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}\lambda}}|\vec{k}\lambda\rangle\langle\vec{k}\lambda|$$

assume $\langle 0|\varphi(0)|0\rangle = 0 \quad \Rightarrow$

$$\langle 0|\varphi(x)\varphi(0)|0\rangle = \sum_\lambda \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}\lambda}} \underbrace{\langle 0|\varphi(0)|\vec{k}\lambda\rangle e^{-ikx}\langle\vec{k}\lambda|\varphi(0)|0\rangle}_{=|\langle 0|\varphi(0)|\vec{k}\lambda\rangle|^2 e^{-ikx}}\Big|_{k^0=E_{\vec{k}\lambda}}$$

for any $\vec{k}, \lambda$ $\exists$ LT Λ with $|\vec{k}\lambda\rangle = \mathcal{U}(\Lambda)\,|\vec{0}\lambda\rangle$

assume $|0\rangle$ to be Lorentz invariant $\mathcal{U}(\Lambda)|0\rangle = |0\rangle$

$$\langle 0|\varphi(0)|\vec{k}\lambda\rangle = \langle 0|\underbrace{\mathcal{U}^\dagger(\Lambda)\varphi(0)\mathcal{U}(\Lambda)}_{=\varphi(\Lambda^{-1}0)\,=\,\varphi(0)}|\vec{0}\lambda\rangle = \langle 0|\varphi(0)|\vec{0}\lambda\rangle$$

$\Rightarrow$

$$\langle 0|\varphi(x)\varphi(0)|0\rangle = \sum_\lambda |\langle 0|\varphi(0)|\vec{0}\lambda\rangle|^2 \int \frac{d^3k}{(2\pi)^3}\frac{e^{-ikx}}{2E_{\vec{k}\lambda}}$$

for free fields we had

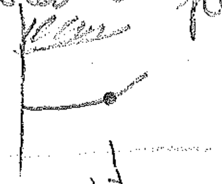
$$\langle 0|\varphi(x)\varphi(0)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}e^{-ikx}$$

which gave $\quad \Delta_F(k) = \dfrac{i}{k_0^2 - E_{\vec{k}}^2 + i\varepsilon} \quad \Rightarrow$

now we obtain

$$\Delta_F(k) = \sum_\lambda \frac{i}{k_0^2 - E_{\vec{k}\lambda}^2 + i\varepsilon}\,|\langle 0|\varphi(0)|\vec{0}\lambda\rangle|^2$$

In general λ is continuous, and $\Delta_F(k)$ has cuts. If there is a mass gap, however, there is an isolated point in the spectrum (for given $\vec{k}$)



This gives a pole

near such a pole:

$$\Delta_F(k) = \frac{iZ}{k^2 - m^2 + i\varepsilon} + \text{non-singular terms}$$

with

$$\sqrt{Z} := \langle 0 | \varphi(0) | \vec{0} \rangle$$

$|\vec{0}\rangle$: 1-particle state with zero momentum

(one can choose the phase of $|\vec{0}\rangle$ such that $\sqrt{Z} \geq 0$)

(*) $\boxed{\text{If } Z \neq 0, \text{ the 2-point function } \Delta_F(k) \text{ in an interacting QFT with a mass gap has a pole at } k^2 = m^2, \text{ were } m \text{ is the physical particle mass.}}$

Z is called <u>wave function renormalization</u>.

remarks:

(i) without interactions $\sqrt{Z} = 1$

(ii) In the derivation of (*) we did not use the fact that φ is the field in $\mathcal{L}$. (*) is also valid for 2-point functions composite operators like, e.g., φ^2.

(iii) In some theories, like QCD, all particles like are 'bound states', and they only show up as poles of composite operators.

2.5 In and out States

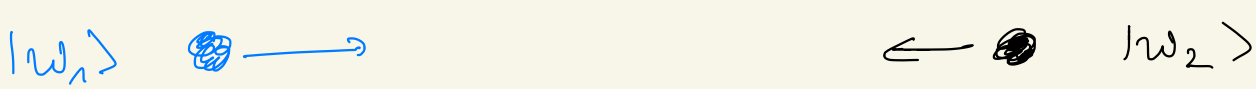
In scattering experiments all particles are far away from each other for $t \to \pm\infty$. If the interaction has finite range, the particles then behave like in 1-particle states.

Here it is important that the particles are somewhat localized. Thus they cannot be in momentum eigenstates. Instead they must be in wave packets

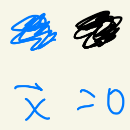
for 1 particle: $\quad |w\rangle = \int d^3k \, |\vec{k}\rangle \, w(\vec{k})$

w: momentum space wave function

We now want to describe the scattering of 2 particles. Let $|w_1\rangle$ and $|w_2\rangle$ be 1-particle states which in the far past were moving towards each other

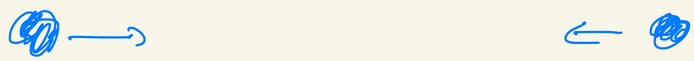


and for $t = 0$ are both localized near $\vec{x} = 0$



To describe the scattering of 2 particles we consider a state $|w_1 w_2 \text{ in}\rangle$ which for $t \to -\infty$ looks like $|w_1\rangle$ in the region where one particle is localized and like $|w_2\rangle$ in the region where the other particle is localized and like $|0\rangle$ otherwise.

$t \to -\infty$:

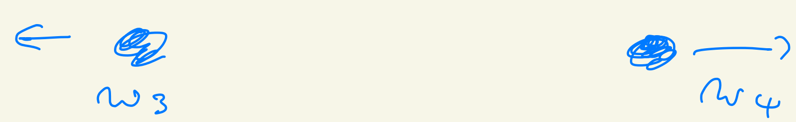
$|w_1 w_2 \text{ in}\rangle$:  <u>in state</u>

If the particles interact, then for $t \to +\infty$ the state $|w_1 w_2 \text{ in}\rangle$ will in general <u>not</u> look like $|w_1\rangle$ in some region and like $|w_2\rangle$ in some other region!

One can have particle production, so that for $t \to \infty$ the state $|w_1 w_2 \text{ in}\rangle$ may not even be a 2-particle state, but instead a superposition of 2-particle, 3-particle, ... states.

Suppose we want to compute the probability amplitude to find exactly 2 particles in wave packets w_3, w_4 for $t \to \infty$. For this define the <u>out state</u> as follows:

$t \to \infty$

$|w_3 w_4 \text{ out}\rangle$: 

Then the amplitude we are looking for is

$$\langle w_3 w_4 \text{ out} | w_1, w_2 \text{ in} \rangle$$

remark: without interactions

$$| w_3 w_4 \text{ out} \rangle = | w_3 w_4 \text{ in} \rangle$$