3 Correlation functions, interactions & scattering

vacuum expectation values of n field operators, called <u>n-point functions</u> or <u>correlation functions</u> or <u>Green's functions</u>, are important objects in QFT.

3.1 Two-point functions

Consider scalar fields. Since operators do not necessarily commute, their ordering plays a role.

Consider: $\langle 0 | \varphi(x_1) \varphi(x_2) | 0 \rangle$.

$|0\rangle$ is invariant under translations $\Rightarrow$

$$\langle 0 | \varphi(x_1)\varphi(x_2) | 0 \rangle = \langle 0 | \varphi(x_1 - x_2) \varphi(0) | 0 \rangle$$

So we can write

$$\Delta^{>}(x_1 - x_2) = \langle 0 | \varphi(x_1) \varphi(x_2) | 0 \rangle$$

This is called a <u>Wightman function</u>.

Define the <u>time ordered product</u> of operators A, B as

$$T A(t_1) B(t_2) := \begin{cases} A(t_1) B(t_2) & \text{for } t_1 > t_2 \\ B(t_2) A(t_1) & t_2 > t_1 \end{cases}$$

$$\Delta_F(x_1 - x_2) := \langle 0 | T \varphi(x_1) \varphi(x_2) | 0 \rangle \quad \text{\underline{Feynman propagator}}$$

$$\Delta_F(x) = \Theta(t) \langle 0 | \varphi(x) \varphi(0) | 0 \rangle + \Theta(-t) \langle 0 | \varphi(0) \varphi(x) | 0 \rangle$$
$$= \Theta(t)\, \Delta^{>}(x) + \Theta(-t)\, \Delta^{>}(-x)$$

Real Klein-Gordon field

Write $\varphi(x) = \varphi_+(x) + \varphi_-(x)$

φ_+: anihilation, φ_-: creation operator

$$\varphi_+(x) = \int \frac{d^3k}{(2\pi)^3\, 2k^0}\, e^{-ik\cdot x}\, a_{\vec k}\Big|_{k^0 = E_k} \qquad \text{write } E_{\vec k} \equiv \sqrt{\vec k^2 + m^2}$$

$$\varphi_-(x) = \left[\varphi_+(x)\right]^\dagger$$

$$\langle 0|\varphi(x)\varphi(0)|0\rangle = \langle 0|\varphi_+(x)\,\varphi_-(x)|0\rangle$$

$$= \int \frac{d^3k_1\, d^3k_2}{(2\pi)^6\, 2k_1^0\, 2k_2^0}\, e^{-ik_1\cdot x}\, \underbrace{\langle 0|a_{\vec k_1} a^\dagger_{\vec k_2}|0\rangle}_{= \langle 0|[\vec a_{k_1}, \vec a_{k_2}]|0\rangle}\Big|_{k_i^0 = E_{\vec k_i}}$$

$$= \int \frac{d^3k}{(2\pi)^3\, 2k^0}\, e^{-ik\cdot x}\Big|_{k^0 = E_{\vec k}}$$

$$\langle 0|\varphi(0)\varphi(x)|0\rangle = \langle 0|\varphi(-x)\varphi(0)|0\rangle \Rightarrow$$

$$\Delta_F(x) = \int \frac{d^3k}{(2\pi)^3}\, \frac{1}{2k^0}\left(\theta(t)\, e^{-ik\cdot x} + \theta(-t)\, e^{ik\cdot x}\right)$$

spatial FT:

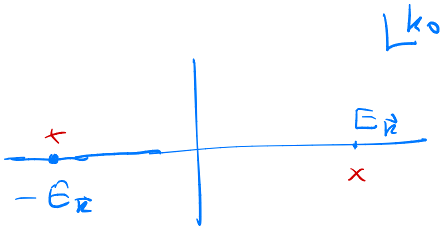
$$\Delta_F(t, \vec k) = \frac{1}{2E_k}\left(\theta(t)\, e^{-iE_{\vec k} t} + \theta(-t)\, e^{iE_{\vec k} t}\right)$$

temporal FT (now with arbitrary $k^0 \in \mathbb{R}$):

$$(*) \quad \boxed{\Delta_F(k) = \frac{i}{k^2 - m^2 + i\varepsilon}} \qquad \text{with } \varepsilon \to 0+ \text{ in the end}$$

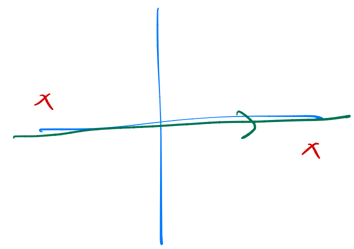
poles of $\Delta_F(k)$: $\quad k_0^2 = \vec{k}^2 + m^2 - i\varepsilon$

$$k_0 = \pm E_{\vec{k}} \mp i\varepsilon$$



Check (*) : $\quad t > 0$

$$\int \frac{dk^0}{2\pi} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik^0 t} =$$



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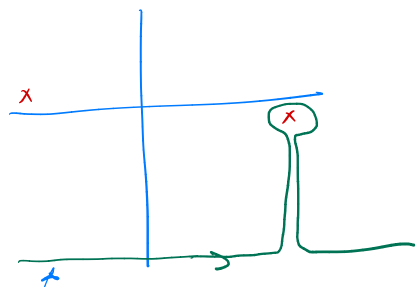

$$= \oint \frac{dk^0}{2\pi} \frac{i}{2E_{\vec{k}}} \frac{e^{-ik^0 t}}{k^0 - E_{\vec{k}}} = \frac{e^{-iE_{\vec{k}} t}}{2E_{\vec{k}}}$$

This goes to zero for $\mathrm{Im}\, k^0 \to -\infty$

similar for $t < 0$ $\quad \square$

$\Delta_F(k)$ has poles near the <u>mass shell</u> $k^2 = m^2$

The $i\varepsilon$ gives the prescription how to integrate around these poles.

The poles are at those k^μ that are 4-momenta of physical particles. Thus the poles of the propagator contain information about the spectrum of the Hamiltonian.

3.2 Interactions

So far we considered Lagrangians quadratic in the fields like

$$\mathcal{L} = \frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2 \right]$$

resulting in linear EOM. Equations for different momenta decouple. The corresponding QFT describes free (i.e. non-interacting) particles.

Now we'll introduce interactions. They have several effects. They lead to interactions between particles and thus to scattering or to bound states. But they also affect single particles, because they change the spectrum of the Hamiltonian. The mass is the energy of a particle at rest, Thus the masses will change, and they no longer equal the constant m appearing in the quadratic term in $\mathcal{L}$. This fact is called <u>mass renormalization</u> (even though m had not been normalized to anything before).

If the EOM is non-linear, different Fourier components get coupled and interact with each other, Non-linear terms in the EOM are obtained by adding higher powers of the fields to $\mathcal{L}$. We want to preserve Lorentz invariance, so the interaction terms in $\mathcal{L}$ should be Lorentz invariant.

Examples: $\varphi^3, \varphi^4, \varphi^5, \dots, (\partial\varphi)^4, \dots$

Mass dimensions

With $\hbar = c = 1$, the action $S = \int d^4x\, \mathcal{L}$ is dimensionless, and x^μ has dimension $(\text{mass})^{-1}$.

This is written as $[S] = 0, \quad [x^\mu] = -1$

$[A]$ is called mass dimension of A

$[d^4x] = -4 \quad \Rightarrow \quad [\mathcal{L}] = 4 \quad , [m^2\varphi^2] = 4 \quad \Rightarrow$

$\boxed{[\varphi] = 1}$

Interaction terms: $\mathcal{L}_{int} = -\frac{g}{3}\varphi^3 - \frac{\lambda}{4}\varphi^4 + \dots$

g, λ : coupling constants

$[g] = 1 \quad , \quad [\lambda] = 0$

The higher the mass dimension of an operator in $\mathcal{L}$, the lower the mass dimension of its coefficient.

Expectation: an operator with mass dimension $n+4$ and coupling constant h contributes like $h\,E^n$ to some power to a process with typical energy E.
⇒ Higher dimensional operators can be neglected at sufficiently small energies.

2.1