2.3 Particles, spin, statistics

Let $|q\rangle$ be a P_ν eigenstate with eigenvalues q_ν,

$$P_\nu |q\rangle = q_\nu |q\rangle$$

Consider the state $a^\dagger_{\vec{k}}|q\rangle$: We found $[P_\nu, a^\dagger_{\vec{k}}] = k_\nu a^\dagger_{\vec{k}} \Rightarrow$

$$P_\nu a^\dagger_{\vec{k}} |q\rangle = ([P_\nu, a^\dagger_{\vec{k}}] + a^\dagger_{\vec{k}} P_\nu)|q\rangle = (k_\nu + q_\nu) a^\dagger_{\vec{k}} |q\rangle$$

$a^\dagger_{\vec{k}}|q\rangle$ is P_ν eigenstate with eigenvalues $k_\nu + q_\nu$
or $a^\dagger_{\vec{k}}|q\rangle = 0$.

Now consider $a_{\vec{k}}|q\rangle$:

$$P_\nu a_{\vec{k}} |q\rangle = ([P_\nu, a_{\vec{k}}] + a_{\vec{k}} P_\nu)|q\rangle = (-k_\nu + q_\nu) a_{\vec{k}} |q\rangle$$

$a_{\vec{k}}|q\rangle$ is P_ν eigenstate with eigenvalue $(-k_\nu + q_\nu)$
or $a_{\vec{k}}|q\rangle = 0$.

$a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ add and subtract 4-momentum k^μ to $|q\rangle$

In particular, they increase and decrease the energy by k^0.

$H = P^0$ is a positive definite operator, its eigenvalues must be bounded from below. $\Rightarrow$

There must be at least one state $|0\rangle$ such that

$$a_{\vec{k}}|0\rangle = 0 \qquad \forall \vec{k}$$

$$|0\rangle = \text{ground state}$$

We can subtract a constant from P^μ such that

$$P^\mu |0\rangle = 0$$

without changing the commutation relations.

$$|\vec{k}\rangle := a^\dagger_{\vec{k}} |0\rangle \quad , \quad P^\mu |\vec{k}\rangle = k^\mu |\vec{k}\rangle$$

$$|\vec{k}_1, \vec{k}_2\rangle := a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} |0\rangle$$

$$P^\mu |\vec{k}_1, \vec{k}_2\rangle = (k_1^\mu + k_2^\mu) |\vec{k}_1, \vec{k}_2\rangle$$

energy = sum of 1-particle energies.

interpretation: $|0\rangle$ no particles, vacuum

$|\vec{k}\rangle$ 1 particle with 4-momentum k^μ

$k^2 = m^2 \Rightarrow$ particle has mass m

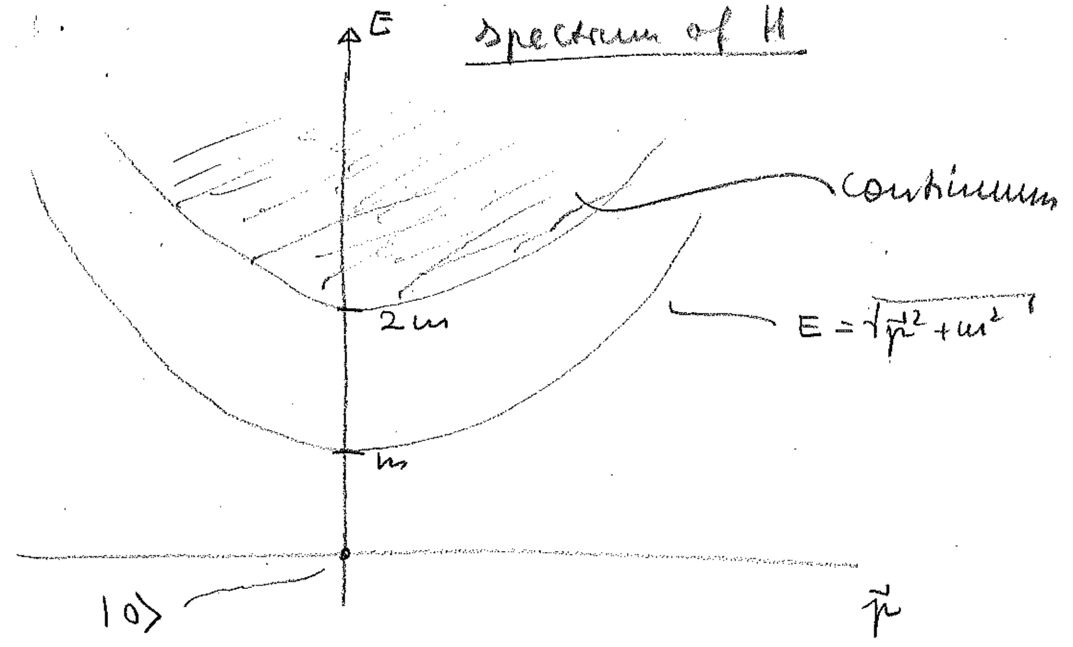
$|\vec{k}_1, \vec{k}_2\rangle$ 2 particles with momenta k_1^μ, k_2^μ

particles are non-interacting

$a^\dagger$ creation operator, a annihilation operator

The Hilbert space contains states with 0, 1, 2, ... particles

<u>Fock space</u>



$$|\vec{k}_2, \vec{k}_1\rangle = a^\dagger_{\vec{k}_2} a^\dagger_{\vec{k}_1} |0\rangle = |\vec{k}_1, \vec{k}_2\rangle$$

symmetric under $(1 \leftrightarrow 2)$ $\Rightarrow$
particles are <u>bosons</u>

<u>Spin</u>

What is the spin of these particles?

$\boxed{\text{Spin = angular momentum in the rest frame.}}$

Conservation of angular momentum $\vec{J}$ follows from invariance under rotations.

Symmetry transformation: $\varphi(\vec{x}) \mapsto \varphi(\underset{\text{rotation matrix}}{R^{-1}}\vec{x})$

infinitesimal rotation:

$$\delta_\varepsilon \varphi = i[\vec{\varepsilon}\cdot\vec{J}, \varphi]$$

$\vec{\varepsilon}$

insert $(*)$, consider only the $\vec{k}=0$ piece.
Since it has no $\vec{x}$-dependence, it does not change at all under rotations. $\Rightarrow$

$$[\vec{J}, a_{\vec{0}}] = 0 \quad , \quad [\vec{J}, a^\dagger_{\vec{0}}] = 0$$

assume $\vec{J}|0\rangle = 0$.

$a^\dagger_{\vec{0}}|0\rangle$ 1 particle at rest

$\vec{J}\, a^\dagger_{\vec{0}}|0\rangle = 0 \Rightarrow$ particle has zero $\vec{J}$

particle has spin 0.

1. Spin and Statistics

In NRQM it is simply an assumption, that identical particles obey either Bose or Fermi statistics.

We have obtained Bose statistics above.

It followed from $[\varphi(\vec{x}), \varphi(\vec{x}')] = [\pi(\vec{x}), \pi(\vec{x}')] = 0$

$$[\varphi(\vec{x}), \pi(\vec{x}')] = i\,\delta(\vec{x}-\vec{x}')$$

Could we have obtained fermions?

We should have used anticommutator instead of commutator, i.e.,

$$\{A,B\} := AB + BA$$

$$\{\varphi(\vec{x}), \varphi(\vec{x}')\} = \{\pi(\vec{x}), \pi(\vec{x}')\} = 0,$$

$$\{\varphi(\vec{x}), \pi(\vec{x}')\} = i\,\delta(\vec{x}-\vec{x}')$$

Then we would have found $\{a^{+}(\vec{p}_1), a^{+}(\vec{p}_2)\} = 0$,

and thus $|\vec{p}_1, \vec{p}_2\rangle = -|\vec{p}_1, \vec{p}_2\rangle$

But now consider the Hamilton operator

$$H = \frac{1}{2}\int d^3x \left\{ \pi^2(\vec{x}) + (\nabla\varphi)^2 + m^2\varphi^2 \right\}$$

now $\pi^2(\vec{x}) = \pi(\vec{x})\pi(\vec{x}) = -\pi(\vec{x})\pi(\vec{x})$

$\Rightarrow \pi^2 = 0$, and also $(\nabla\varphi)^2 = 0$, $\varphi^2 = 0$

That is, the operator H would be identically zero, and this quantum theory does not make much sense.

This is a special version of the spin-statistic theorem: In relativistic QFTs (at least in 4 dimensions), one has no choice whether one uses commutator or anticommutator or whether the particles are bosons or fermions. Instead:

Particles with integer spin are bosons

" " half-" " are fermions

2.4 Quantized complex scalar field

Now consider the complex Klein-Gordon field

(*) $$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2k^0} \left\{ e^{-ikx} a_{\vec{k}} + e^{ikx} b_{\vec{k}}^{+} \right\}_{k^0 = \sqrt{\vec{k}^2 + m^2}}$$

canonical commutator: $[\varphi(\vec{x}), \dot{\varphi}^{+}(\vec{x}')] = i\,\delta(\vec{x} - \vec{x}')$

All other equal time commutators of $\varphi, \dot{\varphi}, \varphi^{+}, \dot{\varphi}^{+}$ vanish. This gives

$$\boxed{[a_{\vec{k}}, a_{\vec{k}'}^{+}] = [b_{\vec{k}}, b_{\vec{k}'}^{+}] = 2k^0 (2\pi)^3 \delta(\vec{k} - \vec{k}')}$$

all other commutators of a, a^{+}, b, b^{+} vanish.

$a_{\vec{k}}^{+}, b_{\vec{k}}^{+}$ ($a_{\vec{k}}, b_{\vec{k}}$) create (annihilate) particles with 4-momentum k^{μ}.

Now we have 2 types of particles: a particles and b particles.

We saw that $\mathcal{L}$ is invariant under $\varphi \mapsto e^{i\varepsilon}\varphi$,

$$\delta_{\varepsilon}\varphi = \varepsilon i \varphi$$

The corresponding Noether charge generates these:

$$\delta_{\varepsilon}\varphi = \varepsilon i [Q, \varphi] \quad \Longrightarrow \quad [Q, \varphi] = \varphi$$

Insert (*) $\Rightarrow$ $[Q, a_{\vec{k}}] = a_{\vec{k}}$, $[Q, b_{\vec{k}}^{+}] = b_{\vec{k}}^{+}$

$\Rightarrow$ $[Q, a_{\vec{k}}^{+}] = -a_{\vec{k}}^{+}$. Let $Q|0\rangle = 0$.

1-particle states $|\vec{k}\, a\rangle := a_{\vec{k}}^{+}|0\rangle$, $|\vec{k}\, b\rangle = b_{\vec{k}}^{+}|0\rangle$

$Q|\vec{k}\, a\rangle = -|\vec{k}\, a\rangle$, $Q|\vec{k}\, b\rangle = |\vec{k}\, b\rangle$

a-particles have charge -1, b-particles $+1$;
b particles are <u>antiparticles</u> of a-particles
particles & antiparticles have equal mass