QFT

2. Scalar field quantization

2.1 Canonical quantization [Srednicki 3]

In QM the commutation relations are obtained by replacing $\{f, g\}_P \to -i[f, g]$

which are now operators and $[\,,\,]$ is the commutator.

These relations are then in the Schrödinger picture. They are also valid in the Heisenberg picture if the operators are taken at equal times:

$$[q_i(t), p_j(t)] = i\delta_{ab}, \ldots$$

In the Heisenberg picture the state vectors are time independent, and the operator EOM are obtained by the same replacement:

$$\dot{f} = i[H, f] + \frac{\partial f}{\partial t}$$

In QFT we will be mostly using the Heisenberg picture:

$$\boxed{\begin{aligned} &[\varphi_a(t, \vec{x}), \pi_b(t, \vec{x}')] = i\,\delta(\vec{x} - \vec{x}')\,\delta_{ab} \\ &[\varphi_a(t, \vec{x}), \varphi_b(t, \vec{x}')] = [\pi_a(t, \vec{x}), \pi_b(t, \vec{x}')] = 0 \end{aligned}}$$

These are the <u>canonical commutation relations</u>.

φ is called <u>field operator</u> or quantum field

real Klein-Gordon field : $\pi = \dot{\varphi}$

complex " $\pi = \dot{\varphi}^*$

In both cases $(\partial^2 + m^2)\varphi = 0$

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2.2 Quantized real Klein-Gordon field

Consider the general solution to the equation of motion, which is a superposition of plane waves. For the real Klein-Gordon field

$$\varphi(x) = \int \frac{d^3p}{2p^0(2\pi)^3}\left\{e^{-ipx}a_{\vec{p}} + e^{ipx}a^+_{\vec{p}}\right\}$$

where $p^0 = \sqrt{\vec{p}^2+m^2}$

note that $\varphi(x)$ is Hermitean, $(\varphi(x))^+ = \varphi(x)$

The factor $\frac{1}{2p^0}$ is a convention, but a useful one: The integration measure can be written as

$$\int \frac{d^3p}{2p^0} = \int d^4p\ \delta(p^2-m^2)\ \Theta(p^0)$$

which is Lorentz invariant.

One can solve (*) for a, a^+: $\quad \varphi(t,\vec{p}) := \int d^3x\, e^{i\vec{p}\vec{x}}\varphi(t,\vec{x})$

$$\varphi(t,\vec{p}) = \frac{1}{2p^0}\left(e^{-ip_0t}a_{\vec{p}} + e^{ip_0t}a^+_{-\vec{p}}\right)$$

$$\dot{\varphi}(t,\vec{p}) = -\frac{i}{2}\left(e^{-ip_0t}a_{\vec{p}} - e^{ip_0t}a^+_{-\vec{p}}\right)$$

$$e^{ip_0t}\left[p_0\varphi(t,\vec{p}) + i\dot{\varphi}(t,\vec{p})\right] = a_{\vec{p}} \quad \Leftrightarrow$$

$$a_{\vec{p}} = \int d^3x\, e^{ipx}\left[p_0\varphi(x) + i\pi(x)\right]$$
$$a^+_{\vec{p}} = \int d^3x\, e^{-ipx}\left[p_0\varphi(x) - i\pi(x)\right]$$

independent of t.

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put $t=0$:

$$[a_{\vec{p}}, a^{+}_{\vec{p}'}] = \int d^3x \int d^3x' \, e^{i(\vec{p}'\cdot\vec{x}' - \vec{p}\cdot\vec{x})}$$

$$\underbrace{[p_0\varphi(\vec{x}) + i\pi(\vec{x}), p_0'\varphi(\vec{x}') - i\pi(\vec{x}')]}_{= (-i)(p_0 + p_0')\, i\, \delta(\vec{x}-\vec{x}')}$$

$$= (p_0 + p_0') \int d^3x \, e^{i(\vec{p}'-\vec{p})\vec{x}} \quad \Rightarrow$$

$$\boxed{[a_{\vec{p}}, a^{+}_{\vec{p}'}] = 2p_0 (2\pi)^3 \delta(\vec{p}-\vec{p}')}$$

similarly one obtains $\boxed{[a_{\vec{p}}, a_{\vec{p}'}] = 0}$

note that by replacing 3 spatial dimensions by 0 spatial dimensions in the above formulas we would obtain the harmonic oscillator with frequency $p_0 = m$. In this case a^+, a are the raising and lowering operators which increase and decrease the energy by m:
If $|n\rangle$ are the energy eigenstates,

$$H|n\rangle = (n+\tfrac{1}{2})m\,|n\rangle \quad (n = 0, 1, 2, \ldots) \text{ then}$$

$$a^+|n\rangle = \text{const}\,|n+1\rangle$$
$$a\,|n\rangle = \text{const}\,|n-1\rangle$$

Something very similar happens in field theory.

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[Nachtmann 3.2]

For the real Klein-Gordon field we had

$$(*)\qquad \varphi(x) = \int \frac{d^3k}{(2\pi)^3 2k^0}\left(e^{-ikx} a_{\vec{k}} + e^{ikx} a^{+}_{\vec{k}}\right), \quad k^0 = \sqrt{\vec{k}^2 + m^2}$$

and $[a_{\vec{k}}, a^{+}_{\vec{k}'}] = 2k^0 (2\pi)^3 \delta(\vec{k} - \vec{k}')$, $[a_{\vec{k}}, a_{\vec{k}'}] = 0$

expectation: a^+, a are raising and lowering operators.

To see this consider space-time translations

$$\varphi'(x) = \varphi(x+\varepsilon) = \varphi(x) + \varepsilon^\nu \partial_\nu \varphi(x),$$

so $\delta_\varepsilon \varphi = \varepsilon^\nu \partial_\nu \varphi$

We use the fact that these are generated by the corresponding conserved charges which are the 4-momentum operators $P_\nu = \int d^3x \, T^0{}_\nu$.
For classical fields we had $\delta_\varepsilon \varphi = -\{\varepsilon \cdot P, \varphi\}_P$. In QFT we have to replace $\{\cdot,\cdot\}_P \to -i[\cdot,\cdot] \Rightarrow$

$$\boxed{\partial_\nu \varphi = i\,[P_\nu, \varphi]}$$

N.B. For $\nu = 0$ this is $\dot{\varphi} = i\,[P_0, \varphi]$ which is just the operator EOM since $P_0 = H$.

Insert $(*)$:

$$\partial_\nu \varphi = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0}\left(-ik_\nu e^{-ikx} a_{\vec{k}} + ik_\nu e^{ikx} a^{+}_{\vec{k}}\right)$$

$$i[P_\nu, \varphi] = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0}\left(e^{-ikx}\, i[P_\nu, a_{\vec{k}}] + e^{ikx}\, i[P_\nu, a^{+}_{\vec{k}}]\right) \Rightarrow$$

$$\boxed{[P_\nu, a^{+}_{\vec{k}}] = k_\nu a^{+}_{\vec{k}}} \quad \text{and } [P_\nu, a_{\vec{k}}] = -k_\nu a_{\vec{k}}$$