1.5 Complex Klein-Gordon field

Consider a complex scalar field φ, $\varphi \neq \varphi^*$ with

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi$$

equation of motion

$$0 = \delta S = \int d^4x \left\{ (\delta \partial_\mu \varphi^*) \partial^\mu \varphi - m^2 (\delta \varphi^*) \varphi + \text{c.c.} \right\}$$

c.c. = complex conjugate

$$= \int d^4x \left\{ -\delta\varphi^* \left[\partial_\mu \partial^\mu + m^2 \right] \varphi - \delta\varphi \left[\partial_\mu \partial^\mu + m^2 \right] \varphi^* \right\}$$

Naively treating the variations $\delta\varphi^*$ and $\delta\varphi$ as independent we would obtain

$$\boxed{(\partial^2 + m^2)\varphi = 0} \qquad \text{(complex) Klein-Gordon eq.}$$

Of course they are not independent, but the result is nevertheless correct: [Coleman QFT lectures]

We have

$$\delta S = \int d^4x \left\{ \delta\varphi^* A + \delta\varphi A^* \right\}$$

We can make purely real variations, so that $\delta\varphi = \delta\varphi^*$ or purely imaginary ones, so that $\delta\varphi = -\delta\varphi^*$.

The first gives $A + A^* = 0$ and the second $A - A^* = 0 \Rightarrow$ $A = A^* = 0$ $\square$

$\mathcal{L}$ is invariant under

$$\varphi \mapsto e^{i\varepsilon}\varphi \quad , \quad \delta_\varepsilon \varphi = i\varepsilon\varphi \, , \quad \chi = i\varphi$$

$$\varphi^* \mapsto e^{-i\varepsilon}\varphi \qquad \delta_\varepsilon \varphi^* = -i\varepsilon\varphi^* , \quad \chi^* = -i\varphi^*$$

This type of symmetry is not related to space-time transformation and is called an inner symmetry. According to Noether's theorem this implies a conserved current. For several fields, the Noether current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a} \chi_a - K^\mu$$

Here we have contributions from $\varphi_a = \varphi, \varphi^*$.

Furthermore, $\delta_\varepsilon \mathcal{L} = 0 \Rightarrow K^\mu = 0$, and

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \chi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^*} \chi^* = \partial^\mu \varphi^* \, i\varphi + \partial^\mu \varphi (-i\varphi^*)$$

$$\boxed{J^\mu = -i \varphi^* \overleftrightarrow{\partial^\mu} \varphi} \qquad \text{with } A\overleftrightarrow{\partial^\mu}B := A\partial^\mu B - (\partial^\mu A)B$$

Conserved charge:

$$\int d^3x \, J^0 = -i \int d^3x \, \varphi^* \overleftrightarrow{\partial_t} \varphi$$

When we quantize this theory we will see that Q counts the total electric charge of the particles in the system.

1.6 Poisson brackets, symmetry generators

Before moving on to quantisation, recall a useful way of writing Hamiltonian mechanics, which is in terms of Poisson brackets. These relations take exactly the same form in QM if one replaces the Poisson bracket by $(-i)$ times the commutator.

In mechanics, the Poisson bracket of two functions $f(t,p,q)$ and $g(t,p,q)$ with $q=(q_1,q_2,\dots)$ is defined as

$$\{f,g\}_P = \frac{\partial f}{\partial q_i}\frac{\partial g}{p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

Properties: $\{f,g\}_P = -\{g,f\}_P$, $\{p_i,p_j\} = \{q_i,q_j\} = 0$

$$\{q_i,p_j\} = \delta_{ij}$$

time evolution of f:

$$\frac{df}{dt} = \{f,H\}_P + \frac{\partial f}{\partial t}$$

All these relations can be generalized to field theory by replacing $i \to (\vec{x},a)$ $\sum_i \to \int d^3x \sum_a$, $\frac{\partial}{\partial q_i} \to \frac{\delta}{\delta \varphi_a(\vec{x})}, \dots$

For simplicity we keep the mechanics notation in this section.

The following observation will be very useful in QFT:

The Noether charge

$$J = \frac{\partial L}{\partial \dot{q}_i} X_i - K = p_i X_i - K$$

of the symmetry transformation $\delta_\varepsilon q_i = \varepsilon X_i$ is the generator of this transformation in the sense that

$$X_i = -\{J, q_i\}_P$$

This is easy to see if Q and K do not depend on p:
Use $\frac{\partial L}{\partial \dot{q}_i} = p_i \Rightarrow -\{J, q_i\}_P = \frac{\partial}{\partial p_i}(p_j X_j - K) = X_i$ <u>o.k.</u>

However, this need not be the case, like for example for $q_i(t) \mapsto q_i(t+\varepsilon)$ where $X_i = \dot{q}_i$.

For the general case we have to re-do the derivation of Noether's theorem in terms of p & q by considering [G. Barnich]

$$L_H = p_i \dot{q}_i - H(t, p, q)$$

and the transformation $\delta_\varepsilon q_i = \varepsilon X_i$, $\delta_\varepsilon p_i = \varepsilon \xi_i$

One can always assume that X & ξ do not depend on time derivatives.

[using the EOM one can replace, e.g., $\dot{q}$ by $\{q, H\}_P$]

$$\delta_\varepsilon L_H = \varepsilon\left(\xi_i \dot{q}_i + \underbrace{p_i \dot{\chi}_i}_{=\frac{d}{dt}(p_i\chi_i) - \dot{p}_i\chi_i} - \frac{\partial H}{\partial p_i}\xi_i - \frac{\partial H}{\partial q_i}\chi_i\right)$$

This is a symmetry if

$$\delta_\varepsilon L_H = \varepsilon \frac{d}{dt} K$$

K cannot contain time derivatives because the LHS contains only first derivatives.

$$\xi_i \dot{q}_i - \dot{p}_i \chi_i - \frac{\partial H}{\partial p_i}\xi_i - \frac{\partial H}{\partial q_i}\chi_i = \frac{d}{dt}\underbrace{(K - p_i\chi_i)}_{=-J}$$

Identify the terms proportional to derivatives $\dot{q}$, $\dot{p}$

$\Rightarrow$

$$\xi_i = -\frac{\partial J}{\partial q_i}, \quad \chi_i = \frac{\partial J}{\partial p_i}$$

Thus we have $\boxed{\xi_i = -\{J, p_i\}_P, \quad \chi_i = -\{J, q_i\}_P}$ □