1.3 Hamiltonian formulation

mechanics: canonical momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$

Hamiltonian $H = -L + p_i \dot{q}_i$

EOM: $\dot{q}_i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

For fields we have the "continuous index" $\vec{x}$. We have to use functional derivatives defined through

(i) $\frac{\delta}{\delta \varphi_a(\vec{x})} \varphi_b(\vec{x}') = \delta_{ab}\, \delta(\vec{x} - \vec{x}')$

(ii) $\frac{\delta}{\delta \varphi_a(\vec{x})}$ satisfies the product rule

Then the canonical momenta are

$$\pi_a(\vec{x}) := \frac{\delta L}{\delta \dot{\varphi}_a(\vec{x})}$$

and

$$H = -L + \int d^3x\, \pi_a(\vec{x})\, \dot{\varphi}_a(\vec{x})$$

eqs. of motion: $\dot{\varphi}_a(\vec{x}) = \frac{\delta H}{\delta \pi_a(\vec{x})}$, $\dot{\pi}_a(\vec{x}) = -\frac{\delta H}{\delta \varphi_a(\vec{x})}$

We consider $L = \int d^3x\, \mathcal{L}(\varphi, \partial\varphi, x)$

$$\pi_a(\vec{x}) = \int d^3x' \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_b} \frac{\delta \dot{\varphi}_b(\vec{x}')}{\delta \dot{\varphi}_a(\vec{x})} = \int d^3x' \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_b} \delta_{ab}\, \delta(\vec{x} - \vec{x}') \Rightarrow$$

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_a}$$

Hamiltonian $H = \int d^3x\, \mathcal{H}$ with

$\mathcal{H} = -\mathcal{L} + \pi_a \dot{\varphi}_a$ Hamiltonian density

1.4 Noether's theorem [E. Noether 1918] [Peskin, Schroeder 2.2]

(*) $\boxed{\text{Every continuous symmetry of the action implies the conservation of some current density and vice versa}}$

A very beautiful and important insight for physics!

Since the symmetry is continuous, we may consider infinitesimal transformations

(**) $$\varphi_a \mapsto \varphi_a' = \varphi_a + \varepsilon \chi_a[\varphi]$$

with an infinitesimal parameter ε.

Write $\delta_\varepsilon \varphi \equiv \varphi' - \varphi$. (**) induces a change

$$\delta_\varepsilon \mathcal{L} = \mathcal{L}(\varphi', \partial\varphi') - \mathcal{L}(\varphi, \partial\varphi)$$

We call (**) a symmetry transformation if

$$\delta_\varepsilon \mathcal{L} = \varepsilon\, \partial_\mu K^\mu \quad \text{for any field } \varphi.$$

implying that if $\varphi(x)$ is a solution to the EOM, then φ' is a solution as well.

Check: $$\delta S' = \delta(S + \delta_\varepsilon S) = \delta(\delta_\varepsilon S) = \int d^4x\, \delta(\delta_\varepsilon \mathcal{L})$$
$$= \varepsilon \int d^4x\, \partial_\mu \delta K^\mu$$

We assume $\delta\varphi = 0$ and thus $\delta K^\mu = 0$ outside a bounded region.
Then Gauss' law implies $\delta S' = 0$ $\square$

N.B. here we did not assume $\delta_\varepsilon S = \varepsilon \int d^4x\, \partial_\mu K^\mu$ to vanish.

Now to the proof of (*) : " $\Longrightarrow$ "

$$\delta_\varepsilon \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \varphi_a} \delta_\varepsilon \varphi_a + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a} \delta_\varepsilon \partial_\mu \varphi_a$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi_a} \delta_\varepsilon \varphi_a + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a} \delta_\varepsilon \varphi_a \right] - \delta_\varepsilon \varphi_a \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a} = \varepsilon \partial_\mu K^\mu$$

Let φ_a satisfy the EOM $\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a}$. Then

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a} \delta_\varepsilon \varphi_a \right] = \varepsilon \partial_\mu K^\mu . \quad \text{Use } \delta_\varepsilon \varphi_a = \varepsilon X_a \Longrightarrow$$

$$\boxed{\partial_\mu J^\mu = 0 \qquad \text{with} \qquad J^\mu \equiv \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a} X_a - K^\mu}$$

$$\partial_\mu J^\mu = \partial_t J^0 + \nabla \cdot \vec{J} \qquad \text{continuity equation}$$

" $\Longleftarrow$ " Assume there is a J^μ with $\partial_\mu J^\mu = \left(\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a} \right) G_a$

such that $\partial_\mu J^\mu$ for solutions to the EOM

Let $\delta_\varepsilon \varphi_a = \varepsilon G_a \Longrightarrow$

$$\delta_\varepsilon \mathcal{L} = \underbrace{\frac{\partial \mathcal{L}}{\partial \varphi_a} \varepsilon G_a}_{\stackrel{!}{=} \varepsilon \left(\partial_\mu J^\mu + G_a \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a} \right)} + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a} \varepsilon \partial_\mu G_a = \varepsilon \partial_\mu \underbrace{\left(J^\mu + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi_a} G_a \right)}_{\equiv K^\mu} \qquad \square$$

If the fields fall off sufficiently fast at infinity, then the charge

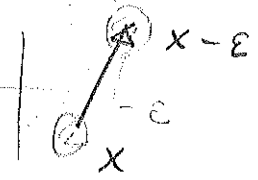
$$Q \equiv \int d^3x \, J^0$$

is well defined. It is time independent (conserved)

$$\dot{Q} = \int d^3x \, \partial_t J^0 = - \int d^3x \, \nabla \cdot \vec{J} = 0 \quad \text{by Gauss' law.}$$

Energy - momentum conservation

If $\mathcal{L}$ has no explicit x-dependence, the space-time translations



$$x^\mu \mapsto x^\mu - \varepsilon^\mu$$

give the symmetry transformations

$$\varphi'(x) = \varphi(x+\varepsilon)$$

We have 4 independent transformations, corresponding to 4 parameters ε^ν.

$$\delta_\varepsilon \varphi = \varphi(x+\varepsilon) - \varphi(x) = \varepsilon^\nu \partial_\nu \varphi$$

So we have 4 X's, labelled by ν:

$$\boxed{X_\nu = \partial_\nu \varphi}$$

$$\delta_\varepsilon \mathcal{L} = \mathcal{L}(\varphi(x+\varepsilon), \partial\varphi(x+\varepsilon)) - \mathcal{L}(\varphi(x), \partial\varphi(x))$$
$$= \varepsilon^\mu \partial_\mu \mathcal{L}(\varphi(x), \partial\varphi(x))$$

For 1 parameter we had $\delta_\varepsilon \mathcal{L} = \varepsilon \partial_\mu K^\mu$. Now we have 4 K^μ's, labelled by ν:

$$K^\mu{}_\nu = \delta^\mu{}_\nu \mathcal{L}$$

$\Rightarrow$ 4 conserved currents, labelled by ν

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} X_\nu - K^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\nu \varphi - \delta^\mu{}_\nu \mathcal{L}$$

or

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial^\nu \varphi - \eta^{\mu\nu} \mathcal{L}}$$ energy - momentum tensor

is conserved $\boxed{\partial_\mu T^{\mu\nu} = 0}$

conserved "charge":

$$\int d^3x\, T^{00} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot\varphi} \dot\varphi - \mathcal{L} \right) = H$$

H = Hamiltonian = energy = P^0 0th component of P^μ, the 4-momentum of the field system.

all conserved "charges":

$$\underline{P^\nu} = \int d^3x\, T^{0\nu}$$

$$P^n = \int d^3x\, T^{0n} \quad \text{spatial momentum}$$

For $\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi)$:

$$P^n = \int d^3x\, \partial^0 \varphi\, \partial^n \varphi = -\int d^3x\, \partial_0 \varphi\, \partial_n \varphi \quad \text{or}$$

$$\vec{P} = -\int d^3x\, \dot\varphi\, \nabla \varphi$$

T^{m0} : energy current density

T^{mn} : momentum current density

} flowing in in direction m

component of momentum