4/24

Why quantum field theory (QFT)?

- There are fields in nature: electromagnetic (EM), gravitational

They should obey the rules of quantum mechanics (QM)

Example: electromagnetic wave

Maxwell's eqs (Heaviside-Lorentz):

$$\nabla\cdot\vec{E} = \rho \qquad \nabla\times\vec{B} - \frac{1}{c}\dot{\vec{E}} = \vec{j}$$

$$\nabla\times\vec{E} + \frac{1}{c}\dot{\vec{B}} = 0 \qquad \nabla\cdot\vec{B} = 0$$

without charges: $\rho = 0$, $\vec{j} = 0$:

$$\left.\begin{array}{l} \nabla\times\nabla\times\vec{E} = \nabla\underbrace{\nabla\cdot\vec{E}}_{=0} - \Delta\vec{E} \\ \frac{1}{c}\nabla\times\dot{\vec{B}} = \frac{1}{c}\partial_t\frac{1}{c}\dot{\vec{E}} \end{array}\right\} \Rightarrow \left(\frac{1}{c^2}\partial_t^2 - \Delta\right)\vec{E} = 0$$

Fourier transform: $\vec{E}(t,\vec{k}) = \int d^3x\, e^{-i\vec{k}\vec{x}}\,\vec{E}(t,\vec{x}) \Rightarrow$

$$\left(\partial_t^2 + c^2\vec{k}^2\right)\vec{E}(t,\vec{k}) = 0$$

For each $\vec{k}$ we have a harmonic oscillator (HO)*, with

frequency $\omega = c|\vec{k}|$.

QM: their possible energies are $E_n = (n+\frac{1}{2})\hbar\omega$

We call these energy eigenstate n-photon states.

Photons are quanta of the EM field. They have particle-like properties.

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* more precisely: $\mathrm{Re}[\vec{E}(t,\vec{k})]$ and $\mathrm{Im}[\vec{E}(t,\vec{k})]$ each describe two HOs for the two transverse polarisations. The momenta of the corresponding photons are $\mp\vec{k}$.

Remarkable: <u>all</u> known particles can be understood as quanta of some fields!

(in the Standard Model of particle physics)

- nature is relativistic

S. Weinberg "QFT [...] is the only way to reconcile the principles of QM [...] with those of special relativity."

- QFT is a powerful tool many-particle systems, also non-relativistic ones.

1 Classical field theory, symmetries

1.1 Principle of least action

reminder: classical mechanics with 1 degree of freedom, coordinate q.

Lagrangian $L(q, \dot{q}, t)$

action $S[q] := \int dt \; L(q(t), \dot{q}(t), t)$

equation of motion follows from the principle of least action

$\delta S = 0$ with $\delta S = S[q + \delta q] - S[q]$, linearized in δq.

multiple degrees of freedom: coordinates $q_i(t)$

generalization for fields: $q_i(t) \to \varphi(t, \vec{x})$

The role of the discrete index i is taken over by the continuous index $\vec{x}$. φ represents infinitely many degrees of freedom, one for each point in space $\vec{x}$.

In general we will have several fields φ_a with a discrete index $a = 1, \dots, N$

Example: electric field $\vec{E}$ has $N=3$ vector components E_i

action: $S[\varphi] = \int dt \; L[\varphi, \dot{\varphi}, t]$

We assume that L is local, meaning that

$$L[\varphi, \dot{\varphi}, t] = \int d^3x \; \mathcal{L}(\varphi(t, \vec{x}), \dot{\varphi}(t, \vec{x}), \nabla\varphi(t, \vec{x}), \dots, t, \vec{x})$$

where the Lagrangian density $\mathcal{L}$ only depends on a finite number of derivatives of φ.

example: (*) $\mathcal{L} = \frac{1}{2}\left[\dot{\varphi}^2 - (\nabla\varphi)^2 - m^2\varphi^2\right]$, $\varphi \in \mathbb{R}$

Notation: space-time coordinates x^μ, $\mu \in \{0,1,2,3\}$

$x^0 = t$, $\vec{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$, $\frac{\partial \varphi_a}{\partial x^\mu} := \partial_\mu \varphi_a$

$$S = \int d^4x\, \mathcal{L}$$

Specialize to $\mathcal{L} = \mathcal{L}(\varphi, \partial\varphi, x)$

equation of motion (EOM)

$$\delta S = S[\varphi + \delta\varphi] - S[\varphi]$$

$$= \int d^4x \{\mathcal{L}(\varphi + \delta\varphi, \partial\varphi + \delta\partial\varphi, x) - \mathcal{L}(\varphi, \partial\varphi, x)\}$$

$$= \int d^4x \left\{\frac{\partial\mathcal{L}}{\partial\varphi_a}\delta\varphi_a + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_a}\underbrace{\delta\partial_\mu\varphi_a}_{=\partial_\mu\delta\varphi_a}\right\}$$ (Einstein convention)

$$= \underbrace{\int d^4x\, \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_a}\delta\varphi_a\right]}_{=0} + \int d^4x\, \delta\varphi_a\left(\frac{\partial\mathcal{L}}{\partial\varphi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_a}\right)$$

$=0$ assuming that $\delta\varphi \neq 0$ only inside a bounded region

$\delta S = 0$ for any infinitesimal $\delta\varphi \Rightarrow$

$$\boxed{\frac{\partial\mathcal{L}}{\partial\varphi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi_a} = 0}$$ Euler-Lagrange equation

For our example (*): $\frac{\partial\mathcal{L}}{\partial\varphi} = m^2\varphi$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} = \partial_t\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} + \partial_m\frac{\partial\mathcal{L}}{\partial\partial_m\varphi} = \ddot{\varphi} - \Delta\varphi$$

EOM: $m^2\varphi - \ddot{\varphi} + \Delta\varphi = 0$ (real) Klein-Gordon equation

QFT

Lorentz invariance

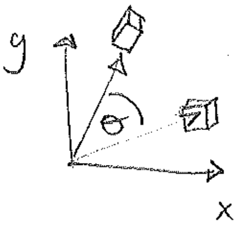
space-time coordinates of an event x^μ $(\mu = 0,1,2,3)$

$x^0 = t$ $(c=1)$ $\quad x^m$ $(m = 1,2,3)$ cartesian coordinates

Oftentimes Lorentz transformations (L.T.) are considered to be passive, that is, the coordinate system is transformed and the physical system is left unchanged.

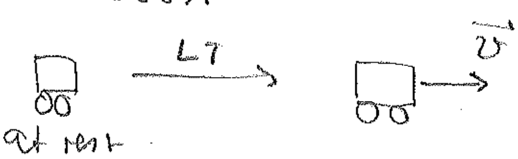
We use the <u>active</u> point of view, the physics is transformed

Examples (i)



rotation

(ii) Lorentz boost



coordinates of the transformed event (Einstein summation conv.)

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \text{or} \quad x' = \Lambda x$$

with $x = \begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix}$, $\Lambda = (\Lambda^\mu{}_\nu)$, 4×4 matrix

Any quantity transforming like x is called a 4-vector

example: 4 momentum p^μ $\qquad p^0$ = energy

$\qquad\qquad\qquad\qquad\qquad \vec{p}$ = spatial momentum

LT leave scalar product

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$$

invariant, $a' \cdot b' = a \cdot b$.

Def. $(\eta_{\mu\nu}) := \operatorname{diag}(1, -1, -1, -1)$ <u>metric tensor</u>. Then

$$a \cdot b = \eta_{\mu\nu} a^\mu b^\nu$$

$$a' \cdot b' = \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma a^\rho b^\sigma = a^\rho (\Lambda^T \eta \Lambda)_{\rho\sigma} b^\sigma$$
$$= a \cdot b \quad \forall a, b \iff$$

$$\boxed{\Lambda^T \eta \Lambda = \eta}$$

Consequence: $(\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1$

LT which are continuously connected to $\mathbb{1}$ have $\det \Lambda = 1$

examples: rotations, Lorentz boosts

examples with $\det \Lambda = -1$: time reversal $t \to -t$

space reflection $\vec{x} \to -\vec{x}$

Def. $x_\mu := \eta_{\mu\nu} x^\nu$. Then

$$a \cdot b = a_\mu b^\mu$$

<u>inverse metric</u>: $\eta^{\mu\nu} := \eta_{\mu\nu}$, $\eta^{\mu\nu} \eta_{\nu\rho} = \delta^\mu{}_\rho$

A second rank tensor $T^{\mu\nu}$ transforms like

$$T'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}$$

example: electromagnetic field strength tensor $F^{\mu\nu}$