I. FERMION PATH INTEGRATION - PHYSICS 571 - WINTER 2014

A. The Fermionic Harmonic Oscillator

The prototype for a fermion field is a 2-state system in quantum mechanics, with ground state $|0\rangle$ and excited state $|1\rangle$, which in QFT will correspond to a fermion state being unoccupied or occupied respectively. I will refer to this as the "fermionic harmonic oscillator" because we can write the Hamiltonian as

$$H = \tfrac{1}{2}m\left(b^\dagger b - bb^\dagger\right) = m\left(b^\dagger b - \tfrac{1}{2}\right) \ , \tag{1}$$

where b and $b^\dagger$ are operators satisfying anti-commutation relations

$$\{b,b\} = \left\{b^\dagger, b^\dagger\right\} = 0 \ , \qquad \left\{b^\dagger, b\right\} = 1 \ , \tag{2}$$

where $\{A,B\} \equiv AB + BA$. The normalized eigenstates of H consist of the ground state $|0\rangle$ which is annihilated by b:

$$b|0\rangle = 0 \ , \tag{3}$$

and the excited state

$$|1\rangle = b^\dagger|0\rangle \ , \tag{4}$$

satisfying

$$H|0\rangle = -\tfrac{1}{2}m|0\rangle \ , \qquad H|1\rangle = +\tfrac{1}{2}m|1\rangle \ . \tag{5}$$

B. Coherent States

It is convenient to introduce the "coherent states"

$$|\psi\rangle = e^{-\bar\psi\psi/2}\left(|0\rangle + \psi|1\rangle\right) \ , \qquad \langle\bar\psi| = e^{-\bar\psi\psi/2}\left(\langle 0| + \langle 1|\bar\psi\right) \tag{6}$$

where the independent Grassmann numbers ψ and $\bar\psi$ which are anti-commuting:

$$\{\psi,\psi\} = \left\{\bar\psi,\psi\right\} = \left\{\bar\psi,\bar\psi\right\} = 0 \ . \tag{7}$$

Note that these are numbers, not Hilbert space operators, and they *commute* with b and $b^\dagger$. Using the anti-commuting property of Grassmann numbers, it follows that

$$\begin{aligned} |\psi\rangle &= \left(1 - \tfrac{1}{2}\bar\psi\psi\right)|0\rangle + \psi|1\rangle \ , \\ \langle\bar\psi| &= \langle 0|\left(1 - \tfrac{1}{2}\bar\psi\psi\right) + \langle 1|\bar\psi \ . \end{aligned} \tag{8}$$

Using the nature of Grassmann numbers, you should be able to show that these states – called coherent states – obey the following useful properties:

$$b|\psi\rangle = \psi|\psi\rangle \ , \tag{9}$$

$$\langle\bar\psi|b^\dagger = \langle\bar\psi|\bar\psi \tag{10}$$

$$\langle\bar\psi_1|\psi_2\rangle = e^{-\frac{1}{2}\bar\psi_1\psi_1 - \frac{1}{2}\bar\psi_2\psi_2 + \bar\psi_1\psi_2} \ , \tag{11}$$

$$\langle\bar\psi|\psi\rangle = 1 \ , \tag{12}$$

$$|\psi\rangle\langle\bar\psi| = (1 - \bar\psi\psi)|0\rangle\langle 0| + \bar\psi|0\rangle\langle 1| + \psi|1\rangle\langle 0| - \bar\psi\psi|1\rangle\langle 1| \ . \tag{13}$$

C. Completeness and Grassmann integration

What remains to establish is a completeness relation. We define Grassmann integration so that

$$\int d\bar{\psi}\, d\psi\, |\psi\rangle\langle\bar{\psi}| = \mathbf{1} = |0\rangle\langle 0| + |1\rangle\langle 1|\ . \tag{14}$$

From eq. (14) we see that integration is therefore defined to look like derivation:

$$\int d\bar{\psi}\, d\psi = \partial_{\bar{\psi}}\partial_{\psi}\ , \tag{15}$$

where derivatives with respect to a Grassmann number are themselves Grassmann...in particular, $\{\partial_{\bar{\psi}}, \partial_{\psi}\} = \{\partial_{\psi}, \overline{\psi}\} = 0$. You should check that this counterintuitive definition gives the correct result, that having $\partial_{\bar{\psi}}\partial_{\psi}$ act on the expression in eq. (13) gives the desired result on the left hand side of eq. (14).

Consider an general function

$$F(\psi) = f_0 + \psi f_1\ . \tag{16}$$

If F is an ordinary number, then f_0 is a number and f_1 is a Grassmann number that anticommutes with ψ (I will assume this, but keep in mind that in supersymmetry you will occasionally encounter a Grassmann function F in which case f_0 is Grassmann and f_1 is an ordinary number). Note that with f_1 being Grassmann, the order makes a difference: $\psi f_1 = -f_1\psi$.

Then we have

$$\int d\psi\, F(\psi) = f_1\ , \tag{17}$$

For a function of both ψ and $\bar{\psi}$ we have

$$F(\psi, \bar{\psi}) \equiv f_0 + \psi f_1 + \bar{\psi} f_2 + \bar{\psi}\psi f_3\ , \qquad \int d\bar{\psi}\, d\psi\, F(\psi, \bar{\psi}) = -f_3\ . \tag{18}$$

where f_3 is an ordinary number if F is.

D. Grassmann Path Integration

Now suppose you want to construct

$$Z = \langle\overline{\psi}_f|e^{-iH(t_f - t_i)}|\psi_i\rangle\ , \qquad \psi_i \equiv \psi(t_i)\ , \quad \psi_f \equiv \psi(t_f) \tag{19}$$

as a path integral. We break of the time interval $T = (t_f - t_i)$ into a lot of small pieces $T = N dt$ with

$$\psi(t_i + ndt) \equiv \psi_n\ , \qquad \psi_0 = \psi(t_i)\ , \qquad \psi_N = \psi(t_f)\ , \tag{20}$$

and similarly for $\bar{\psi}$, and then we use the completeness relation for coherent states in eq. (14) to write

$$Z = \int d\bar{\psi}_1 d\psi_1 \cdots d\bar{\psi}_{N-1} d\psi_{n-1}\, \langle\bar{\psi}_N|e^{-iH\,dt}|\psi_{N-1}\rangle\langle\bar{\psi}_{N-1}|e^{-iHdt}|\psi_{N-2}\rangle \cdots \langle\bar{\psi}_1|e^{-iH\,dt}|\psi_0\rangle \tag{21}$$

A typical term in the integrand is of the form (dropping the zero-point energy)

$$\begin{aligned}
\langle\bar{\psi}_n|e^{-iH\,dt}|\psi_{n-1}\rangle &= \langle\bar{\psi}_n|e^{-imb^\dagger b\,dt}|\psi_{n-1}\rangle \\
&= \langle\bar{\psi}_n|1 - imb^\dagger b\,dt + O(dt^2)|\psi_{n-1}\rangle \\
&= \left(1 - im\bar{\psi}_n\psi_{n-1}\,dt + O(dt^2)\right)\langle\bar{\psi}_n|\psi_{n-1}\rangle \\
&= e^{\left(-im\bar{\psi}_n\psi_{n-1}dt - \frac{1}{2}\bar{\psi}_n\psi_n - \frac{1}{2}\bar{\psi}_{n-1}\psi_{n-1} + \bar{\psi}_n\psi_{n-1}\right)}\ ,
\end{aligned} \tag{22}$$

where in the last line follows from eq. (11).

Replacing the ψ_n by a continuous function of t the above expression may be written as

$$\langle\bar{\psi}_n|e^{-iH\,dt}|\psi_{n-1}\rangle \;=\; e^{idt\bar{\psi}(t)(\frac{i}{2}\overleftrightarrow{\partial}_t-m)\psi(t)+O(dt^2)} \tag{23}$$

Taking the product of all the terms in eq. (21) and taking the limit $dt\to 0$ yields the path integral

$$Z=\int D\bar{\psi}D\psi\, e^{iS}\ , \qquad S=\int dt\,\bar{\psi}(\tfrac{i}{2}\overleftrightarrow{\partial}_t-m)\psi\ , \tag{24}$$

with boundary conditions $\psi(t_i)=\psi_i$, $\psi(t_f)=\psi_f$. If we take $\psi_i=\psi_f=0$ we can integrate by parts in S and obtain

$$S=\int dt\,\bar{\psi}(i\partial_t-m)\psi\ . \tag{25}$$

Note that we can define correlation functions of the form

$$\langle T(\psi(t_1)\cdots\psi(t_k)\bar{\psi}(t_{k+1})\cdots\bar{\psi}(t_n))\rangle=\frac{1}{Z}\int D\bar{\psi}D\psi\, e^{iS}\psi(t_1)\cdots\psi(t_k)\bar{\psi}(t_{k+1})\cdots\bar{\psi}(t_n)\ . \tag{26}$$

The fact that this gives the *time ordered* correlation function is easy to see by going back to the discrete variables $\psi_1...\psi_N$.

E. Generalization to Dirac fermions in four dimensions

The generalization of the fermionic path integral above to free Dirac fermions four dimensions is straight forward: We just replace the Grassmann numbers ψ and $\bar{\psi}$ by Grassmann 4-component spinors, and replace S by the Dirac action,

$$Z=\int D\bar{\psi}D\psi\, e^{iS_D}\ , \qquad S_D=\int d^4x\,\bar{\psi}(i\partial\!\!\!/-m)\psi\ . \tag{27}$$

F. Performing Grassmann Path Integrals

Suppose we have Grassmann field ψ and $\bar{\psi}$ and the Grassmann integral

$$Z=\int D\bar{\psi}D\psi\, e^{-\int\bar{\psi}\mathcal{D}\psi} \tag{28}$$

where CD is some hermitian differential operator with orthonormal eigenstates χ_n and eigenvalues λ_n:

$$\mathcal{D}\chi_n=\lambda_n\chi_n\ . \tag{29}$$

Then we can expand ψ and $\bar{\psi}$ in terms of these eigenstates:

$$\psi(x)=\sum_n c_n\chi_n(x)\ , \qquad \bar{\psi}(x)=\sum_n \bar{c}_n\chi_n^\dagger(x)\ , \tag{30}$$

where the $\chi_n(x)$ are ordinary functions, while the c_n and $\bar{c}_n$ are independent Grassmann numbers. Then the path integral becomes

$$\begin{aligned} Z &= \int\prod_n d\bar{c}_n\,dc_n\, e^{-\sum_{m,n}\lambda_n\bar{c}_m c_n\int d^4x\,\chi_m^\dagger(x)\chi_n(x)} \\ &= \int\prod_n d\bar{c}_n\,dc_n\, e^{-\sum_n\lambda_n\bar{c}_n c_n} \\ &= \int\prod_n d\bar{c}_n\,dc_n\,\prod_n(1-\lambda_n\bar{c}_n c_n) \\ &= \prod_n\lambda_n\ . \end{aligned} \tag{31}$$

But this is nothing other than the determinant of the operator $\mathcal{D}$, so we have

$$Z = \int D\bar{\psi}D\psi\, e^{-\int \bar{\psi}\mathcal{D}\psi} = \det \mathcal{D} \ . \tag{32}$$

Note the difference between this and gaussian path integration over bosonic variables ϕ and ϕ^*:

$$\int D\phi^* D\phi\, e^{-\int \phi^*\mathcal{D}\phi} \propto \frac{1}{\det \mathcal{D}} \tag{33}$$

where an uninteresting overall normalization is neglected.

When thinking about fermionic path integrals it is important to remember that the canonical fields $\bar{\psi}$ and ψ obey nontrivial equal time commutation relations, while the path integral variables $\bar{\psi}$ and ψ are Grassmann fields, not operators, and all anticommute with each other:

$$\{\psi(x), \psi(y)\} = \{\bar{\psi}(x), \psi(y)\} = \{\psi(x), \bar{\psi}(y)\} = \{\bar{\psi}(x), \bar{\psi}(y)\} = 0 \ . \tag{34}$$

G. Including sources

We can generalize the partition function for free Dirac fermions by adding Grassmann sources for the fermion field. Defining $\mathcal{D} = (i\partial\!\!\!/ - m)$ we have

$$\begin{aligned} Z(\eta,\bar{\eta}) &= \int D\bar{\psi}D\psi\, e^{i\int d^4x\, \bar{\psi}\mathcal{D}\psi + \bar{\eta}\psi + \bar{\psi}\eta} \\ &= \int D\bar{\psi}D\psi\, e^{i\int d^4x\, \left(\bar{\psi}\psi + \bar{\eta}\mathcal{D}^{-1}\right)\mathcal{D}\left(\mathcal{D}^{-1}\eta + \psi\right) - \bar{\eta}\mathcal{D}^{-1}\eta} \\ &= e^{-i\int d^4x\, \bar{\eta}\mathcal{D}^{-1}\eta} \times Z(0,0) \end{aligned} \tag{35}$$

where the last equality is obtained by shifting the dummy integration variables to $\bar{\psi}' = \left(\bar{\psi} + \bar{\eta}\mathcal{D}^{-1}\right)$ and $\psi' = \left(\mathcal{D}^{-1}\eta + \psi\right)$.

It follows that correlation functions are given by

$$\langle T(\psi(x_1)\cdots\bar{\psi}(x_{k+1})\cdots\rangle = \frac{1}{Z(0,0)}\left[\left(-i\frac{\delta}{\delta\bar{\eta}(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta\eta(x_{k+1})}\right)\cdots\right] Z(\eta,\bar{\eta})\bigg|_{\eta=\bar{\eta}=0} \tag{36}$$

In particular, the propagator is given by

$$\langle T\psi(x_1)\bar{\psi}(x_2)\rangle = \frac{i}{\mathcal{D}}_{x_1,x_2} \tag{37}$$