Problem C5.1 Apply the Klein-Gordon operator $\partial^2 + m^2$ directly to $\Delta_F(x) = \langle 0|T\varphi(x)\varphi(0)|0\rangle$ to show that for a free real scalar field

$$(\partial^2 + m^2)\Delta_F(x) = -i\delta^4(x).$$

Note that the time derivatives in the Klein-Gordon operator can act on either the field or the time-ordering step functions.

Problem H5.1 In Problem C2.1 you saw that an infinitesimal Lorentz transformation can be written as $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$. and that the coefficients $\omega_{\mu\nu} \equiv \eta_{\mu\rho}\omega^\rho{}_\nu$ are antisymmetric, $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

(a) Lorentz invariance implies that there are conserved 'charges' $J^{\mu\nu}$ such that

$$U^\dagger(\Lambda)\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x)$$

with

$$U(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)$$

Show that

$$[J^{\mu\nu}, \varphi(x)] = -M^{\mu\nu}\varphi(x)$$

with

$$M^{\mu\nu} := i(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

(b) Verify the commutation relation

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho})$$

Problem H5.2

(a) Consider 2 particles with collinear velocities $\mathbf{v}_1$, $\mathbf{v}_2$. Show that

$$E_1E_2|\mathbf{v}_1 - \mathbf{v}_2| = \left[(p_1\cdot p_2)^2 - m_1^2m_2^2\right]^{1/2}$$

Hint: First check this in the rest frame of particle 2. Then perform a Lorentz boost in the direction of the momentum of particle 1.

(b) Use the result from (a) to write the relation between the invariant matrix element and differential cross section in a Lorentz-invariant form.