Problem C2.1

(a) Write an infinitesimal Lorentz transformation as $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$ Show that $\omega^{\mu\nu} \equiv \eta^{\nu\rho}\omega^{\mu}{}_{\rho}$ is antisymmetric, $\omega^{\mu\nu} = -\omega^{\nu\mu}$.

(b) Which components of ω are non-vanishing for (i) a spatial rotation, and (ii) for a Lorentz boost?

(c) Show that the infinitesimal change of a scalar field φ under this transformation can be written as

$$\delta_\omega \varphi = \frac{1}{2}\omega^{\mu\nu}(x_\mu \partial_\nu \varphi - x_\nu \partial_\mu \varphi)$$

(d) Let the Lagrangian density $\mathcal{L}$ be Lorentz invariant. Show that

$$\delta_\omega \mathcal{L} = \frac{1}{2}\omega^{\rho\sigma}\partial_\mu K^{\mu}{}_{\rho\sigma}$$

and determine $K^{\mu}{}_{\rho\sigma}$. Keep in mind that due to the antisymmetry of ω, $K^{\mu}{}_{\rho\sigma}$ should be antisymmetric under $\rho \leftrightarrow \sigma$.

(e) Determine the currents which are conserved according to Noether's theorem. You should find

$$M^{\mu}{}_{\rho\sigma} = x_\rho T^{\mu}{}_{\sigma} - x_\sigma T^{\mu}{}_{\rho}$$

(f) What is the physical meaning of the components $M^{\mu}{}_{rs}$ with $r, s \in \{1, 2, 3\}$? To convince yourself, apply the formula from (e) to a point particle with momentum $\mathbf{p}$ located at position $\mathbf{a}$.

Problem H2.1

(a) Show that the modified energy momentum tensor

$$\Theta^{\mu\nu} := T^{\mu\nu} + \partial_\lambda W^{\lambda\mu\nu}$$

where $W^{\lambda\mu\nu}$ is antisymmetric in its first two indices, $W^{\lambda\mu\nu} = -W^{\mu\lambda\nu}$, is also divergenceless, $\partial_\mu \Theta^{\mu\nu} = 0$, and that it gives the same conserved energy and momentum as $T^{\mu\nu}$.

(b) Consider classical electrodynamics with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Construct the energy-momentum tensor $T^{\mu\nu}$ for this theory using the same procedure as in the lecture. Show that the resulting tensor is not symmetric, $T^{\mu\nu} \neq T^{\nu\mu}$.

(c) Show that the construction in (a), with

$$W^{\lambda\mu\nu} = F^{\mu\lambda}A^{\nu}$$

leads to an energy-momentum tensor $\Theta^{\mu\nu}$ which is symmetric and yields the standard formulas for the electromagnetic energy and momentum densities

$$\epsilon = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2), \qquad \mathbf{S} = \mathbf{E} \times \mathbf{B}$$

Problem H2.2 Finish what is left of problem C2.1.