Problem C1.1 The electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be written in terms of the scalar potential Φ and the 3-dimensional vector potential $\mathbf{A}$ as $\mathbf{E} = -\nabla\Phi - \dot{\mathbf{A}}$, $\mathbf{B} = \nabla \times \mathbf{A}$. One can combine the potentials to form a 4-dimensional vector potential by defining $A^0 = \Phi$. The field strenght tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Verify that the components of the electric and magnetic fields can be written as

$$E_i = F^{i0}, \qquad B_i = \frac{1}{2}\epsilon_{imn}F^{mn}.$$

Here ϵ_{imn} is the the (totally antisymmetric) Levi-Civita symbol with $\epsilon_{123} = 1$.

(b) Show that Maxwell's equations can be written as

$$\partial_\mu F^{\mu\nu} = J^\nu, \qquad \partial_\mu \widetilde{F}^{\mu\nu} = 0$$

Here

$$J = \begin{pmatrix} \rho \\ \mathbf{J} \end{pmatrix}$$

is the 4-vector current density with the charge density ρ and the 3-dimensional current density $\mathbf{J}$. Furthermore, $\widetilde{F}$ is the *dual* field strength

$$\widetilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

with the 4-dimensional Levi-Civita symbol with $\epsilon^{0123} = 1$.

Problem H1.1

(a) Write the Lagrangian density for the electromagnetic field (see problem C1.1)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

in terms of (i) the A^μ and also (ii) in terms of $\mathbf{E}$ and $\mathbf{B}$.

(b) Apply the principle of least action to

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu\right)$$

by varying A^μ and determine the resulting equations of motion. Write them (i) in terms of the A_μ and (ii) in terms of $\mathbf{E}$ and $\mathbf{B}$.

Problem H1.2 Consider a Lorentz transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ with $\eta^{\rho\sigma}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta^{\mu\nu}$. If a field φ transforms as $\varphi \to \varphi'$ with $\varphi'(x) = \varphi(\Lambda^{-1}x)$ it is called a scalar field.

(a) Compute $\partial_\mu\varphi'$. Show that $\partial_\mu\varphi\partial^\mu\varphi$ transforms like a scalar field.

(b) Show that the action $S[\varphi] = \int d^4x \mathcal{L}(\varphi, \partial\varphi)$ with

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)$$

is invariant under this transformation, that is, that $S[\varphi'] = S[\varphi]$.